

An Ample Approach to Data and Modeling

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Abstract

Developments in Science and technology have relied extensively on modeling and pattern recognition. In the present work, we describe a framework for modeling how models can be built that integrates concepts and methods from a wide range of fields. The information schism between the information in the real-world and that which can be gathered and considered by any individual information processing agent is characterized and discussed, which is followed by the presentation of a series of the adopted requisites while developing the reported modeling approach. The issue of mapping from datasets into models is subsequently addressed, as well as some of the main respectively implied difficulties and limitations. Based on these considerations, an approach to meta modeling how models are built is then progressively developed. The reference M^* meta model framework is presented first, which relies critically in associating whole datasets and respective models in terms of a strict bijective association. Among the interesting features of this model are its ability to bridge the gap between data and modeling, as well as paving the way to a paired algebra of both data and models which can be employed to combine models in hierarchical manner. After illustrating the M^* model in terms of patterns derived from regular lattices, the reported modeling approach continues by discussing how sampling issues, error and overlooked data can be addressed, leading to the $M^{<\epsilon>}$ variant. The frequent and important situation in which the data needs to be represented in terms of respective probability densities is treated next, yielding the $M^{<\sigma>}$ meta model, which is then illustrated respectively to a real-world dataset (iris flowers data). Several considerations about how the developed framework can provide insights about data clustering, complexity, causality, network science, collaborative research, deep learning, and creativity are then presented, followed by overall conclusions.

‘So much closer on the lake, the new star.’

LdaFC.

Keywords: Scientific method, modeling, pattern recognition, decision theory, logic, databases, data science, philosophy of science, linguistics, cladistics, taxonomies, neuronal networks, learning, deep learning, reasoning, ontologies, epistemology, cognition, artificial intelligence, complexity, network science, creativity, collaborative science, causality.

1 Introduction

Despite the many intricacies that characterize our world and the universe, there are some properties that seem to underlie in a more shared, fundamental, and systematic manner the structure and dynamics of natural phenomena. Among them, we have the two following intriguing facts: (i) every portion of our world is strongly intercon-

nected and interdependent one another along time and space; and (ii) at the same time, there are *severe* limitations to the information that can be collected and processed by any computing system, be it natural (e.g. humans) or artificial (e.g. digital computers).

Perhaps necessary and unavoidably, these two important principles directly oppose one another, establishing an interesting mutual global/local duality or tension which may have well been, as will be suggested in the present work, at the very core of the appearance and unfolding of life and intelligence themselves. On one side we have a vastness of global interconnections extending through an impressive range of scales, on the other the extremely limited resources of every kind available to any known information processing system, including living beings. This intriguing duality is here deemed here to important and inexorable enough as to warrant the name of *information schism* which, though related to the semantic gap, implies a wider context.

The level of interdependency of natural structures and

phenomena is so breathtaking as not to be often realized. The following two examples should suffice to materialize this issue. First, consider the dynamics of a pendulum on a table in front of us. Because of the gravitational force always established between any two bodies of mass, and as a consequence of this force fields extending to infinity, the movement of this pendulum will be affected by *every single bit of mass in the universe*. Though this influence will certainly become smaller and smaller as the distance between the masses increases, it will nevertheless be present and therefore influence, even as an infinitesimal element in the background noise, in limiting full accuracy to be achieved for the measurements and predictions about the pendulum dynamics. Our second example, which is related to interactions along time, consists of the relatively smaller and localized events that reverberate through lasting periods of time, as it was possibly the case of the meteor that led to dinosaurs being extinct, therefore allowing mammals to evolve.

Interestingly enough, both the above issues become particularly critical in non-linear systems — which encompass virtually a wide range of real-world phenomena — because these can amplify even minute perturbations into major effects. Other critical limitations of information processing by agents include, but are not limited to, the fact that several objects and states are non-observable in nature, the limited available *time* for any action, as well as the possible presence of error and noise in every measurement taken from the real world.

So it is that nature has somehow self-organized itself into a never ending web of interconnections and interdependences extending along every possible scales of space and time. From the perspective of nature itself, this does not constitute a problem, because the physical world does not seem to require help of external computing resources.

However, with the appearance of individual agents, such as living beings, the information schism acquired much more relevance, representing a major challenge to be circumvented in some manner. The main problem here is that agents — be it natural or artificial, individual or collective — can only exchange and process minute portions of mass, energy and information. As the survival and perpetuation of these agents rely critically on their interaction with the environment (which may also include other agents), so as to make suitable decisions with basis on predictions, it becomes mandatory that any of these agents incorporates the adequate means for receiving, processing, predicting, and acting on the environment.

When contemplated from the above discussed perspective, the continuing existence of living beings and other information processing entities can be realized as being truly phenomenal. After some additional reflection, we may perceive that the success of individual living beings

has largely relied on taking timely decisions on what to do based on information sampled from the respective environment, while also taking into account previous experiences consolidated through some kind of memory such as nervous systems or biochemical dynamics characteristic of each species. Figure 1 illustrates the basic condition of an individual A in an environment E .

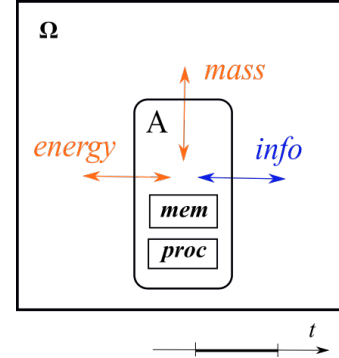


Figure 1: An individual agent A in an environment E exchanges and acts upon mass, energy and information. The survival and reproduction of this agent depends critically on taking timely decisions by processing the information received from the environment and then acting suitably on it, while taking in account previous experiences stored in some manner. Agents capable of information processing can be modeled as incorporating memory and processing capabilities, which are both finite and require some physical expense in terms of mass and energy, being consequently limited. There are also constraints regarding the period of time it takes for predicting and making decisions, and it is important to keep in mind that the environment E almost certainly also contains other information processing agents representing potential threat or collaboration.

A more in-depth view of the concept of decision making is particularly critical for our formulation. It should be recalled that the taking of effective decisions relies critically at least on the following aspects: (i) sampling enough information from the environment; (ii) taking into account previous experience, especially in the sense of possessing comprehensive understanding of the environment; and (iii) identifying the situation as corresponding to some instance of already experienced problem; and (iv) having the means for making accurate *predictions* guiding the possible decisions. More informally, models could be understood as magic mirrors reflecting not only the real-world properties, but also their respective consequences.

An intriguing relationship can thus be established between the action of taking a decision and the basic principle underlying science, namely the *construction of models* by using the scientific method (e.g. [1, 2]).

The scientific method also relies importantly on *obtaining quantified information* about the phenomenon of interest, as well as previous related knowledge, in order not only to *better understand* that phenomenon, but also to *make accurate predictions* about it. Thus, in essence, de-

cision making by information processing entities can be directly related to the development of models. Thus we posit that *decision taking* by agents is basically the same as *scientific modeling*, sharing not only the same objectives, but also the mechanisms for achieving results. In a sense, a model can be understood as a logic-mathematic-computational construct involving conditions that need to be satisfied by the observed data.

Interestingly, a further relationship can be established between individual decision taking with the area of *pattern recognition* (e.g. [3, 4, 5]). In this area of great current interest, the main objective is, given a set of properties or features measured from an object, to reach a conclusion about its possible *class* or category. There are two main types of pattern recognition: *supervised* and *non-supervised*, the former being characterized by availability of previous knowledge, examples or prototypes of the existing categories, which are not available in non-supervised classification. In both cases, and especially in the supervised case, a direct parallel can be established between decision taking/ model building with pattern recognition. The obtention of experimental measurements in modeling can be directly associated with the derivation of properties of the entities to be classified, the consideration of previous experience/models is reflected in the information available about the classes, and the obtained prediction can be directly paired with the action of making predictions.

The intrinsic relationship between decision making, pattern recognition and modeling can also be inferred from the evolutionary perspective that endowed humans (as well as other living beings) with these two abilities. The point here is that, in case pattern recognition and modeling are distinct, they would require different neuronal and cognitive respective abilities, which is much more expensive than sharing the same neuronal resources for addressing both these critically important tasks.

We could go much further because several other actions such as urban planning, land management, education, economic and social policies, to name but a few, may also be related with modeling, decision taking, or pattern recognition. Remarkably, also the field of arts can be related to the modeling framework by understanding an art piece with a dataset and the model with the conditions estimated to be necessary for positive respective appreciation and/or impact. Ultimately, it becomes difficult to find a human activity that can not be somehow related to model building and/or pattern recognition.

The above discussed relationship between decision taking, modeling, and pattern recognition probably represents the most important and critical characteristic of the approach discussed in the current work. For at least the following reason: this allows us to incorporate con-

cepts and methods from a wide range of related scientific fields, especially philosophy of science, artificial intelligence (e.g. [6]) and pattern recognition, discrete mathematics, statistics, physics, complex networks, and data science, to name but a few.

Another important feature of the reported approach regards its logic-mathematic-computational formalization of the activity of model building through a *meta-model*, which can allow us to better understand and draw more objective and general results and conclusions concerning the properties, advantages and limitations of model building.

Though new models can be obtained in a never ending number of ways, here we focus on a methodological framework based on logical combinations of the existing models while considering set operations between the respective datasets. More specifically, instead of logically combining models (which can also be done) irrespectively of data, the considered method also allows expressing the dataset of interest as a combination of other existing datasets, then obtaining the sought model as a logical combination of the models that can be associated to the respectively identified datasets. It is shown that, by establishing an bijective association between the datasets of interest and respective models it becomes possible to obtain a bridge between these two domains, with each set operations being used to manipulate datasets becoming bijectively associated with a respective logical operation.

Given the data and model realms, it is possible to start with some data of particular importance and then look for a model, or vice-versa. An interesting asymmetry seems to characterize the development of science through model constructing, residing in the fact that though new models can be obtained by logic, exact combination of the existing models, these models would still need to be associated to some real world data, which corresponds to its respective physical validation. Interestingly, the proposed bijective association between datasets and models avoids this situation, because every combined model will necessarily be associated to a respective dataset.

In spite of its idealizations in several respects, this first meta-model, which is henceforth referred to as the M^* model, provides a sound reference for better understanding more realistic modeling through the progressive incorporation of characteristics such as noise, incomplete sampling and/or characterization, classification errors, etc.

In this work, the basic overall of the M^* framework is also extended to address problems like sampling, error, noise (yielding the $M^{<\epsilon>}$ meta model), and to take into account the stochastic representation of the data in terms of respective probability densities, therefore leading to the $M^{<\sigma>}$ meta model. Both the M^* and $M^{<\sigma>}$ approaches are illustrated by respective case-examples.

To complement work, we discuss how the concepts and methods related to the developed modeling frameworks can provide insights about areas of great current importance including clustering, complexity, collaborative research, deep learning, and creativity.

2 Specifying the Problem

It often happens that the difficulties in developing a solution to a given problem ultimately derive from lack or imprecisions while specifying the respectively sought goals and constraints. Thus, it is reasonable to initiate the development of the approach reported in this work by listing the many requirements and characteristics that were initially specified.

The main objectives and constraints that have been adopted in the currently described approach are listed in the following:

- [R1] - Allow the integration of several related concepts such as modeling, decision making, pattern recognition, etc.;
- [R2] - Integrate the progressive incorporation of knowledge that characterizes scientific advance;
- [R3] - Accommodate the tension between specificity in modeling datasets bijectively and the generality implied by every data element in those sets non-injectively satisfying the same associated model;
- [R4] - Allow the representation of data elements in terms of respective features (measurements), as it is typical in pattern recognition;
- [R5] - Allow the identification of the main challenges in modeling and other related areas;
- [R6] - Provide subsidies for better understanding clustering, complexity, complex networks, ontologies, collaborative science, deep learning and creativity, among other possibilities;
- [R7] - Adhere to both model- and data-driven perspectives;
- [R8] - Lead to an effective modeling methodology that can be eventually automated in software and/or hardware engines;
- [R9] - Be relatively formal but remain nevertheless accessible, while also maintaining good didactic potential;
- [R10] - Allow the incorporation of stochasticity related to dataset and modeling;
- [R11] - Allow the incorporation of the tuning role of parameters in scientific modeling;
- [R12] - Pave the way to compositions of models, in the sense that the modeling results can be feed backed as input or into other modeling systems;
- [R13] - Be as congruent as possible with the human un-

derstanding of modeling and pattern recognition, as well as many of the involved concepts;

[R14] - Account for the fact that modeling and pattern recognition depends on sets of available or selected data, varying in size and generality from a few data elements to the whole physical world;

[R15] - Allow multiple data elements to be queried simultaneously, as motivated by modeling, and still provide good performance when applied to single individuals.

3 Mapping Datasets into Models

As approached in this work, the basic operation in modeling is considered to be the mapping of datasets into respective models. As such, it is important to discuss this operation in more detail, which consists the main objective of this section. More specifically, we will develop a reasoning allowing an *bijective association* to be established between the datasets and the respective models. Recall that, mathematically, an bijective association consists of an binary association of elements belonging to two sets that has the properties of being *reflexive*, *symmetric* and *transitive*. bijective associations are particularly important because they can be understood as implementing a network of *causal* relationships between the several involved components.

For simplicity's sake and for all subsequent purposes in this work, this type of relationship may be understood as establishing a *identity* or bridge between the dataset and model domains.

Other important issues include the understanding of how *parameters* can be accommodated into models, the need to quantify the properties of the data elements into respective *features or properties*, as well as the several possible *types of models*, not to mention the several manners in which models can be progressively developed. Therefore, it is hoped that the concepts and discussions developed in this section contribute a sound basis for building the sought meta model, as well as for identifying and discussing the possible limitations while mapping datasets into models, which will be addressed in the subsequent section. By *meta modeling* it is henceforth understood the endeavor of modeling how models are built and developed.

We start by presenting, in Figure 2, four types of mappings that may take place between the datasets ω_i in the environment E and the respective models in the model framework M .

In Figure 2(a) we have a *non-injective* mapping, in which more than one dataset ω of E are mapped into the same model m in M . Though it can be understood that both these datasets are explained by that model, it is im-

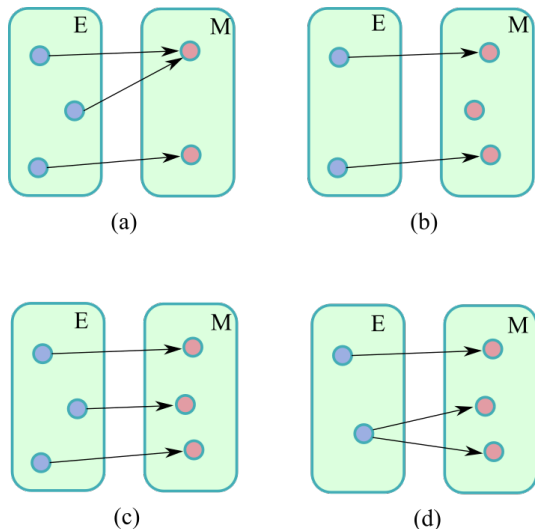


Figure 2: Four possible types of mappings, here understood in the mathematical context, from datasets in E into models in M . (a): *non-injective* mapping, meaning that more than one dataset is associated to a same model. (b) *non-surjective* mapping, in which one model in M is not verified for any dataset in E . (c) The *bijective* situation in which each dataset is associated to a single model, and all models end up associated datasets. The situation in (c) is critically important because it means that the mapping can be inverted. Formally speaking, though the mapping shown in (d) cannot be characterized as a function (though this restriction is relaxed in some approaches), it does represent a relatively common situation while mapping data into models and can be easily addressed by merging those datasets that map into the same model, yielding an injective map.

possible to distinguish between the two original datasets from their respective image. A *non-surjective* mapping is illustrated in Figure 2(b), in which some of the models in M have not been verified respectively to any of the existing datasets in E . This situation could be informally understood as a “model in wait for a dataset”. This situation is also unwanted because we have models that are not verified. The situation depicted in Figure 2(c) corresponds to a *bijective* mapping between the elements of E into M , being therefore *invertible* unlike the two previous situations.

The critical importance of adopting a bijective, invertible mapping of datasets into models resides in the fact that this type of relationship both avoids the ambiguity of a non-injective mappings as well as the existence of unverified models.

A forth situation is worth consideration, and it has to do with mappings that, by not adhering to the usual concept of mathematical function, allow more than one model to be associated to a same dataset, as illustrated in In Figure 2(d). This situation can be addressed by understanding that those multiply satisfied models actually correspond to the same model (giving rise to a subset of features), yielding a respective injective map. As we

will see, this type of mapping is also relatively common regarding data elements, but cannot occur when an bijective association is to be established between datasets and models. This situation may also be caused by insufficient sampling of data or errors.

Interestingly, while in our approach the mapping between datasets and models is henceforth understood as an bijective association, all the *data elements* inside each dataset ω_i map in a non-injective manner into the same model m_i , being therefore *not* subjected to an bijective association. At the same time, any data element may belong to more than one dataset.

The fact that greater freedom of mapping is allowed in the case of data elements is actually welcomed because it disentangles the seemingly opposite requirements in modeling and pattern recognition respectively to having specificity of models regarding whole sets of data, but generalization of models with respect to individual data elements. From the pattern recognition perspective, it means that all datasets in a given ω_i belong to the same category defined by the respective model m_i , which seems to be quite reasonable.

Scientific models can be understood as involving *variables*, *constants*, and *parameters*, among other possible components. Variables include all quantities that may vary during an experiment; constants refer to quantities that never vary during or between experiments; and parameters are quantities that may vary from an experiment to another. Variables are often subdivided as being dependent and independent (or free). As implied by its name, a variable is said to be dependent in case it is expressed in terms of the others in a given model. It is interesting to observe that the concept of variable dependence is relative to each specific model, because a variable that is dependent in one case may be independent in another model.

In the case of a simple pendulum, we have time as a free variable and the angular position and speed as variables dependent of time. The mass of the bob and the length of the rod correspond to parameters. The gravity acceleration can very probably be taken as a constant, given that it is difficult to change its value in a laboratory.

These three main types of modeling elements can be immediately associated with pattern recognition concepts: variables are the measurements (or features) of the data; constants are constants, while parameters correspond to adjustments influencing the measurements or decisions. For instance, in a neuronal network the parameters would correspond to the weights and bias of each neuron, or could refer to the smoothing level adopted while simplifying images. In a physical model, the parameter tuning allow a specific phenomenon to fit the respective model.

The proper setting of parameters is critical for model-

ing and pattern recognition, since they directly influence the decisions. In the present work, we understand that the parameters are always adjusted so as to guarantee the bijective association between datasets and respective models. This adjustment can be made through some *optimization* procedure, varying the parameter values so as to lead to no errors in the decision. The more frequent situation of possible decision errors will be addressed in Section 4. It is also possible to search for suitable parameters value during data analysis and model building.

So far, we have assumed that the data elements in each dataset can be directly operated by the model in order to verify the respective adherence. However, the practical analysis of a given dataset by a model is, in general, impossible unless the elements in this dataset have been first properly represented in terms of a set of categorical or quantitative *features*, also called measurements, characteristics, attributes, and properties, in the pattern recognition area. Therefore, a *further level of mapping* needs to be incorporated into modeling and pattern recognition, extending from datasets into feature sets.

The diagram in Figure 3 illustrates how a given dataset ω_i can be mapped into respective features f_j , $j = 1, 2, \dots, m$, which are then sent to the respective model m_i .

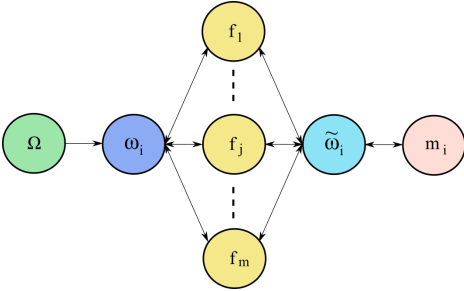


Figure 3: The modeling or recognition of a dataset ω_i derived from a respective universe Ω requires each data element to be first mapped into a set of categorical and/or quantitative features f_j , $j = 1, 2, \dots, m$, which defines in a bijective manner a new dataset $\tilde{\omega}_i$. For simplicity's sake, the feature-associated datasets $\tilde{\omega}_i$ will not be shown in the other diagrams in this work.

Observe that the description of a dataset ω_i therefore give rise to a transformed version of that dataset $\tilde{\omega}_i$, upon which the models can now objectively operate. For simplicity's sake, the latter type of datasets will be omitted from other figures and diagrams of this work, but they will be nevertheless understood to be present.

It is not often realized that features are *always* present in typical decision, modeling and recognition tasks, including the measurements representing the very entities of interest. For instance, before we can decide that the presented entity is or not a dog, it needs to be transformed into an image by our visual systems, or typically

transformed into respective matrices in an artificial system. Features can be transformed and combined in endless manners, but the results can always be understood as features.

As we learn from the pattern recognition area, it often constitutes quite a challenge to select a proper set of features describing the analyzed entities. Observe that the number of features involved in a model defines a respective multidimensional *feature space*. Ideally, each data element should be mapped into a common feature space in a bijective manner, so as to establish an bijective association between the data elements and their representations in terms of the considered features. It is also particularly important to identify the *smallest* set of features that may allow a problem to be reasonably solved. In Figure 3, though all features have been connected through bijective arrows, it may also be the case that each of the features alone is not bijective, but that the set of all considered features will provide a bijective association.

The choice and number of features required for establishing a bijective mapping between data and models also depends on the existing data as well as on the intrinsic characteristics and level of heterogeneity among the data elements. For instance, in case the existing elements are very similar, more features will be required. Alternatively, more heterogenous datasets tend to favor fewer features.

Though the features are assumed to provide a complete, invertible representation of the datasets in our first approach (M^*), which is necessary to maintain the bijective association between data elements and respective models, it is also possible to subsequently adapt this same model for situations when the features no longer provide an invertible mapping with the data elements. For consistency of modeling, we also assume that all data elements in E are always characterized in terms of every considered and applicable features. Observe that the fact of a data element not allowing the derivation of a feature considered in a given existing model m_i may automatically eliminate the possibility of that element satisfying that model, in case the dataset cannot be described in terms of other features. At the same time, a feature could be missed that represents the only manner to discriminate between two distinct datasets.

We have so far addressed several points related to the data elements, datasets, types of mapping of the latter into models, and features. Now, we approach the modeling level itself.

It is important to keep in mind that any model can be immediately associated with a decision or categorization, namely that of the dataset satisfying or not the model, or to which an extent it adheres to the dataset. There are several types of possible models/decisions: thresholds, rules, equations, descriptions, etc. Any of these may be

involved in the henceforth considered modeling.

It is also interesting to divide the possible models into two major groups: (a) those that seek an *optimal* (minimum or maximum of some merit figure); and those aimed at achieving a given property within a reasonable margin of accuracy. While the former type of problems is directly related to the ample and important area of mathematical optimization, the latter involves defining some margin of tolerance and working with probabilities.

Merit or fitness figures can be associated to each obtained model, reflecting the requirements specific to each problem. Possible merit figures include the length of the model description, its intelligibility to humans, and the cost of checking if a dataset satisfies a model, among many other possibilities. A particularly interesting objective is, given a new dataset, to find the largest dataset entirely containing it, as this would account for the most general explanation of that dataset. In this case the larger dataset will nevertheless have to be restricted if one wants to keep the bijective association. Also, unless the smaller dataset has some special significance, it could be therefore subsumed into the larger model. Several such simplifications and specifications are allowed by the proposed meta modeling approach. Another particularly interesting situation concerns, given a set of datasets to find interrelationships between them.

4 Limitations in Mapping Datasets into Models

Several components of our meta-modeling approach have been presented and discussed in the previous section. Now, we address some of the most common types of limitations and constraints related to those components.

Given a dataset ω_i of interest, it is possible one or more its data elements to have been assigned by mistake, or that other data elements be missing. In these cases we will have a dataset that does not fully correspond to our expectations. Let's illustrate this situation in terms of the following example. Let ω be a dataset that has been singled out for modeling as a consequence of having its data elements associated with a posited new plant species. Spurious samples from other species may be included in ω , while other samples of the considered species are overlooked, e.g. by some sampling procedure. These situations will imply in inconsistencies leading to incorrect model being identified for that dataset, and probably lead to less accurate and incomplete model identification and combination.

Missing data elements are characteristic of the sampling that is unavoidably required in case of infinite or too large sets of data elements.

Errors may also occur while mapping datasets into features. These may include mistakes, finite resolution or noise while measuring the features. It is also possible that the equations or program used to estimate the features is intrinsically incorrect, leading to improper characterization. Errors taking place while measuring or calculating features can severely impact the identification of a valid model for the given dataset. Another related problem concerns the fact that a feature that is critically necessary for obtaining a model for a given dataset is overlooked or unknown.

Another possible source of errors takes place at the modeling level itself. Here, we may have inconsistent decisions defined in terms of the features, logical errors, or the overlooking of some important features.

An important type of error not often realized in modeling and pattern recognition is the situation in which some of the data elements in the data environment E have not yet been checked respectively to every existing model, which may also undermine the obtained results.

5 The M* Meta-Model

Having discussed some of the main aspects and components involved in mapping from datasets into models, as well as possible respective limitations, we are now in position of developing a more principled and relatively formal meta-model that can account for as many of the requirements listed in Section 2 as possible.

We start with the overall structure depicted in Figure 4, involving a finite number of possible data elements represented as a universe set Ω . These basic data elements x_i are henceforth assumed to be finite, with $j = -1, 2, \dots, N_\Omega$. Observe that the largest possible environment E corresponds to the *power set* of Ω , containing $2^{(N_\Omega)}$ subsets.

The observable, or available, or restricted set of subsets of Ω are understood to constitute the environment E upon which models can be built, being therefore accessible as a set of datasets ω_i , $i = 1, 2, \dots, N_\omega$, each of which are therefore composed by data elements. The initial configuration of a given model framework can be associated to the respectively assumed *postulates* or *hypothesis*.

The data elements in Ω can be equiprobable (exist in the same number) or not. Interestingly, both cases can be identically addressed by the proposed framework, though the non-equiprobable case will imply in some data elements to be less likely (taking a long time) to be incorporated into E . Probabilistic situations can be approached by using the $M^{<\sigma>}$ framework to be described in a subsequent section.

Observe that a same data element may appear in more

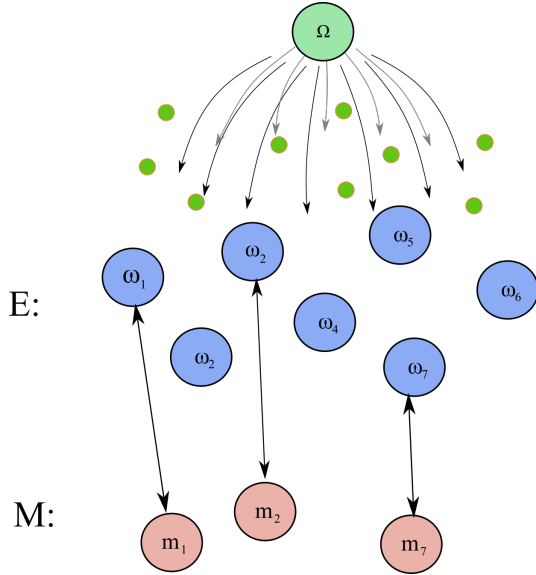


Figure 4: The M^* meta model, decision taking, and pattern recognition. The universe set Ω contains all possible data elements (small green circles) x_j , which can be successively drawn into environmental datasets ω_i , $i = 1, 2, \dots, N_\omega$, so that $\omega_i \subset \Omega$, which define the current data environment E . Each of these datasets may eventually become associated to a respective model m_i explaining necessarily every possible element of ω_i . The set of existing models is understood to correspond to the modeling framework M . The adherence between the data elements in the available datasets and the existing models needs to be continuously updated in order to ensure overall consistency. The critically important bijective association between datasets and models precludes one dataset of being associated to more than a model, and vice versa, but this case can also be easily accommodated into the M^* framework.

than one dataset, as it individually may satisfy more than one model. This property is reasonable and compatible with our concept of modeling and recognition, because a same entity can indeed satisfies several models. For instance, a cat is a mammal, but also a mammal, and it has a tail. Each of these decisions are normally taken as valid categories, though we may be particularly interested in some more specific or general property.

The basic component of the M^* approach, which is henceforth called *cartouche*, is illustrated in Figure 5.

Observe that the cartouche concept encapsulates a basic important principle in the scientific method, that a model is valid until it is found not to work for just a single experiment (data element). The cartouche also intrinsically integrates the two basic domains that are essential for both scientific modeling and pattern recognition, namely datasets and models. From the perspective of scientific modeling, this property emphasizes the importance of theoretical constructions to adhere to existing experimental data, as well as on the essential role of bijective representation of data in terms of features. From the point of view of pattern recognition, emphasis is placed

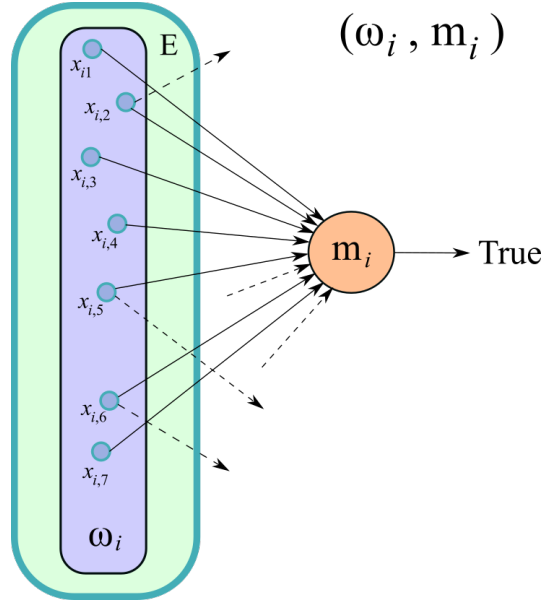


Figure 5: The *cartouche*. Corresponding to the basic component of the M^* approach, the cartouche involves a set of data elements $x_{i,j}$ contained in a dataset ω_i belonging to an dataset environment E , and a respective model m_i . The bijective pairing between the dataset ω_i and its respective model m_i requires that the model is satisfied (True output) whenever *every* data element in ω_i satisfies the logic condition corresponding to the model. Observe that some of the data elements are allowed to map into other models, and that the maps fed from other models into m_i have no effect on the model validity. However, only the model m_i can be associated bijectively with the dataset m_i , implying that the latter cannot map completely into any other model, but if it does so, the two models can be deemed to be equivalent. All the above conditions can be summarized in the mathematical statement (ω_i, m_i) , expressing that there is a bijective association between the dataset ω_i and the respective model m_i , which is therefore understood to fully explain that dataset. Though features level has not been included in this diagram for simplicity's sake, more information about this important level can be found in Section 13.

on the importance of obtaining models for the data.

The bijective association between the current dataset E and the respective modeling framework M can be expressed as (E, M) , being henceforth called the *current theory*. The cartouche can also be immediately extended to reflect that the interconnection between datasets and models is typically accomplished through sets of features F , yielding the triple (E, F, M) .

It is also interesting to defined subsets of models in order to organize distinct modeling approaches. For instance, one can organize the existing models in terms of the mathematical methods (e.g. derivatives, set theory, integrals, delays, etc.) or the type of used features.

Figure 6 presents a zoom of a hypothetical modeling situation, illustrating some of the important features regarding the association between the dataset elements and the respective models.

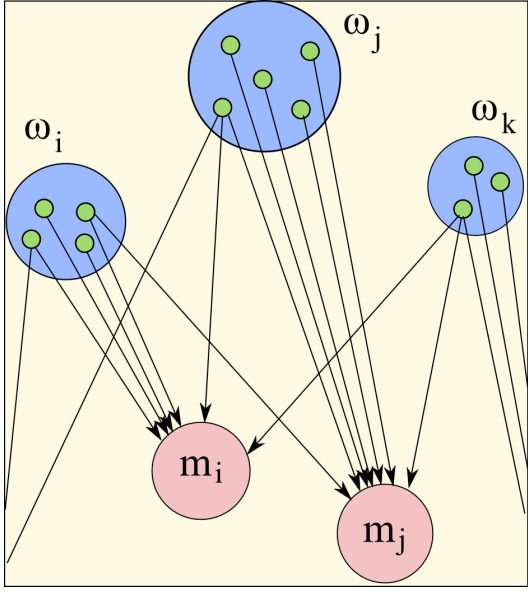


Figure 6: Zooming into an hypothetical situation illustrates the fact that all data elements (green) in a dataset ω_i necessarily map into the same model m_i in a non-injective manner. Each of the arrows extending from a data element x_j to a respective model M_i indicates that x_j satisfies m_i . At the same time, a bijective mapping is ensured between each dataset and its respective model. This feature of the proposed framework is essential for disentangling these two seemingly conflicting characteristics of modeling. Though each of the data elements in a pair (ω_i, m_i) does satisfy the model m_i , this model is only understood to be fully satisfied with respect to all the elements in ω . The figure also shows that a same data element belonging to one of the existing datasets can map into more than one model. The features layer has been omitted for simplicity's sake.

At each instant, the suggested meta model is understood to incorporate N_M models m_i , $i = 1, 2, \dots, N_M$. Each of the datasets ω may become associated to one and only one respective model m , being henceforth understood that *every* element of ω will satisfy the respective model m , and vice-versa, so that an *bijective association* is consequently established between each dataset and the respective model. Observe that this aspect of the M^* framework actually defines two scales or levels of modeling, one at the data element level, and another of higher hierarchy at the dataset level. Also, observe that the mappings from data elements of a dataset ω_i that do not lead to the bijectively paired m_i are not considered at this higher level of associations, but only at the lower data element level. In this way we accommodate the fact that one of the data elements in a dataset may individually satisfy more than one model, but without implying the dataset to which it belongs be assigned to other models. In other words, the association between a dataset and a model takes place only when *all* its data elements satisfy that model.

The current set of available datasets ω_i is henceforth

called *data environment* E , while the existing models are henceforth understood to constitute the *modeling framework* M . The set containing all elements in any of the datasets ω_i of E is henceforth represented as S_E .

Taken jointly, these two sets may be related to the concept of *current knowledge*. Recall that at any time, new data elements can be drawn from Ω and define new datasets that can eventually assume enough importance in order to become subject of respective modeling. Examples of this possibility include but are by no means limited to the appearance of a new species of living beings, the discovery of new stars, the birth of new individuals of a given species, and the invention of new technological devices. As addressed in more detail in Section 6, it is also possible that two or more existing datasets (or pairs dataset-model) be combined through set operations such as union, intersection, complementation or difference.

In order to ensure consistence of proposed framework, the datasets are updated continuously, in the sense that any new element drawn from Ω is checked respectively to each existing model and incorporated into the associated dataset in case it satisfies that model before eventual combination of models can be contemplated. In addition, all the existing data elements are continuously checked respectively to any new incorporated model.

A more complete, expanded representation of the meta model M^* is depicted in Figure 7, also incorporating the *features* associated to each dataset.

Observe that the bijective association between datasets and respectively associated models is maintained by the new feature layer F , being reflected in the confluence of each dataset into a single model even if multiple features were involved in the respective representation. In addition, observe that a same type of feature may be adopted for the characterization of more than one dataset. The incorporated layer consisting of features is henceforth understood to constitute a new layer called the *feature layer* F .

Given a *bijective pairing* (ω_i, m_i) , a decision can be immediately assigned to the model, corresponding to its respective verification by the dataset. Therefore, each model corresponds to a question or decision, a feature that is directly related to the concept of causality.

There are other interesting situations that can also be taken into account while integrating data and models into the current framework. For instance, when a new dataset ω_i is found to be entirely contained in one of the existing datasets ω_j , therefore being a subset of the latter, a *restriction* of ω_j can be assigned to ω_i . For instance, let's say that $\omega_i = \{1, 2, 3, 4, 5, 6, 7\}$ and $\omega_i = \{2, 3, 4\}$, the new dataset can be paired with a respectively restricted model corresponding to the set operation $\omega_i = \omega_j - (\omega_j - \omega_i)$. Interestingly, this sequence of reasoning implies *recursion*,

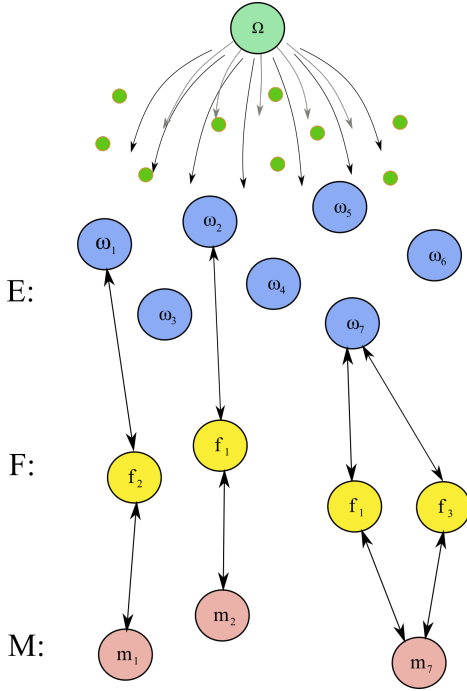


Figure 7: The meta model M^* expanded to incorporate the features respectively describing each of the datasets. It should be recalled that models can only operate on data that has been categoric or quantitatively specified in an objective manner. In addition, more than one dataset be characterized in terms of a same type of feature.

which can be approached by initially assigning a provisory dummy model m_i .

In fact, the association of a model to a dataset can be done as soon as the dataset is given, through a labeling procedure that simply assigns an identifier (without semantic content) to that dataset. Evidently, this type of modeling does not contribute to the modeling framework and cannot lead to a decision being respectively attached to the model other than providing an identified for that dataset.

The above discussion suggests that assigning models to datasets that are subsets of the existing datasets may not only compound the modeling framework, but also contribute to making it to increase in a combinatorial manner. These cases can be more effectively addressed simply by understanding that a new dataset is actually a subset of a larger dataset explained by a more comprehensive model.

Although the basic principle in the M^* is to try to assign a model to every new dataset of interest, cases in which the latter is already fully contained within the existing datasets associated to models would only be justified in case the new datasets has some special relevance requiring its discrimination, through a restriction, from the existing datasets. Otherwise, this type of new dataset can simply be ignored.

The fact that a same data element can map into two or more models motivates the use of measurements to quantify these interrelationships. A possibility consists simply in considering the number $n_{i,j}$ of associations between each data element $x_{i,j}$ in a subset ω and all the existing models. Then, relative frequency histograms (or respective moments) can be used to characterize each dataset (or model). Datasets leading to relatively large average number of multiple connections $\langle n_{i,j} \rangle$ can be understood as being less *specific*, in the sense that the respective data elements are strongly related, though not through bijective relationships, with several models. It is also possible to count the number of individual data element connections received by a given model, as this value provides insights about the generality of the models.

It is equally interesting to study the distribution of these histograms among the several models in the environment E . In case the datasets are found to have similar distributions of the $n_{i,j}$ statistics, the modeling framework can be understood as being more *uniform*.

Observe that the relationship between the datasets and respective models can also be represented in terms of a *bi-partite* network having weights corresponding to the number of respective data elements mapping from a dataset to a model. Such networks can provide valuable information about the overall structure, completeness, malleability, and robustness of the respective modeling framework. It would be of particular interest to devise means for enhancing a given scientific framework while taking into account the topological features of these networks. These networks should also present hierarchical structure reflecting not only the data/model hierarchy, but also how the data elements are distributed amongst the existing models.

An analogue approach can be adopted regarding the types of features in a given E .

While the M^* approach requires the bijective pairing of datasets and models, it is also possible to consider the following situation (and related variations). Given a current dataset environment E and a modeling framework M , it may happen that one of the datasets ω_i already paired with a respective model m_i , defining the pairing (ω_i, m_i) , becomes associated to another distinct model m_j . This corresponds to the mapping situation depicted in Figure 2(d). This is a case of particular interest because, though the mapping from ω_i to m_i is not typically considered a function, the inverse is a function. These situations can be easily accommodated into the M^* approach simply by merging the two models m_i and m_j through their union or intersection, since $(\omega_i \cup \omega_i, m_i \vee m_j)$ and $(\omega_i \cap \omega_i, m_i \wedge m_j)$.

Reaching the most complete knowledge about Ω can be understood as the ultimate goal of modeling. This

situation corresponds to having models associated to every possible subset $\omega \subset \Omega$. Quite interestingly, this can be done in several manners, including the following extreme approach: the model corresponding to each possible subset of Ω consists simply to enumerating its elements. The problem with this trivial solution is that not much is learned about the data elements and their grouping into datasets. Other approaches include the already discussed combination of existing models, as well as developing completely new models based on insights provided by the similarity between datasets by using the index Λ suggested in Section 4.

6 A Paired Algebra of Datasets and Models

Though the M^* meta model has so far been contemplated in a mostly static manner, additional mechanisms may be incorporated allowing the progressive derivation of new models and datasets. A possible respective approach is described in the current section involving either combination of datasets in terms of set operations or the integration of models by using logical connectives, which we henceforth understood as a *paired algebra* of datasets and models. Yet another possibility to be discussed elsewhere is the presentation of new pairs of datasets and models.

It should be also taken into account that the proposed framework can be readily adapted to other types of models, e.g. by using production rules or composition of functions as in neuronal networks. Interestingly, the heuristics usually employed by humans for taking decisions seems to be largely dependent of logical manipulations. This property of the M^* framework is related to the fact that the consistency between models is always guaranteed in terms of the respective data consistency.

Incidentally, the M^* framework and its derivations seems to be largely congruent with the way humans develop models, take decision and perform pattern recognition.

The consistent combination of either datasets or models is immediately allowed by the fact that the pairing between datasets and models corresponds to an bijective association, which establishes a sound bridge between these two important domains. Under these circumstances, it immediately follows that set operations between datasets ω become intrinsically linked to logical manipulations of respective models m . Some examples of the bijective as-

sociations between dataset and m

$$\begin{aligned}
& \text{Dataset Domain } E \iff \text{Model Domain } M \\
& \omega_k = \omega_i \iff m_k = m_i \\
& \omega_k = [\omega_i]^C \iff m_k = \neg m_i \\
& \omega_k = \omega_i \cup \omega_j \iff m_k = m_i \vee m_j \\
& \omega_k = \omega_i \cap \omega_j \iff m_k = m_i \wedge m_j \\
& \omega_k = \omega_i - \omega_j = \omega_i \cap [\omega_j]^C \iff m_k = m_i \wedge \neg m_j \\
& \omega_k = \omega_j - \omega_i = [\omega_i]^C \cap \omega_j \iff m_k = \neg m_i \wedge m_j \\
& \omega_k = [\omega_i \cup \omega_j]^C = [\omega_i]^C \cap [\omega_j]^C \iff m_k = \neg m_i \wedge \neg m_j \\
& \omega_k = [\omega_i \cap \omega_j]^C = [\omega_i]^C \cup [\omega_j]^C \iff m_k = \neg m_i \vee \neg m_j \\
& \omega_k = \omega_i \cup [\omega_j \cap \omega_p] \iff m_k = m_i \vee [m_j \wedge m_p] \\
& \dots
\end{aligned}$$

Where $[]^C$ stands for the set complementation operation respectively to S_E . Observe also that to each set operation does correspond a logical manipulation of models, and vice-versa.

Table 1 presents the 16 possible logical operations between two logical variables m_i and m_j yielding m_k , but out of them 4 are not really useful for obtaining new combinations of models (and datasets): $m_k = \text{TRUE}$, $m_k = \text{FALSE}$, $m_k = m_i$ and $m_k = m_j$.

It is also interesting to observe that it is possible to implement every logical operation in terms of $\neg(x \vee y)$ or $\neg(x \wedge y)$, among other possibilities, which in the data domain becomes $[x \cup y]^C$ and $[x \cap y]^C$, respectively.

It can be shown that there exists a total of 2^n possible logical operations between n logical variables. As a consequence, the number of possible cases steeply increases with the hierarchy of the sets or models. However, the currently available extensive computational resources can be applied, possibly incorporating optimization techniques. At the same time, the incorporation of many hierarchical levels in the description of a newly obtained model also implies that description to become less tangible by human perception, and therefore more *complex* and *abstract*. For these reasons, the efficiency of the modeling framework greatly depends on the choice of models for the initial framework, in the sense that some of these choices may contribute to explaining a new dataset in terms of a relatively (or, ideally, minimal) combination of the previous models.

The subject of obtaining new models through set operations between the existing datasets (or logical between models) is as extensive as it is interesting and cannot not be fully addressed here. However, an interesting approach consists of employing intersection, union, complementation and difference between a small number of datasets.

When translated to the human perspective, this intrinsic combinatorial complexity of possible models becomes

X	Y	op_0	op_1	op_2	op_3	op_4	op_5	op_6	op_7	op_8	op_9	op_{10}	op_{11}	op_{12}	op_{13}	op_{14}	op_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 1: The 16 possible logical operations between two logical (or Boolean) variables X and Y . Remarkably, some of them — such as “not”, “and” , and “or” — are closer to human cognition in the sense of being more frequently employed. Provided the conditions for the bijective association between datasets and models is fulfilled, there will be a set operation respective to each of the 16 logical operations.

closely related to the concept of complexity, because it becomes more and more expensive [7] to develop and understand highly hierarchical models. This provides a motivation for having experts in specific areas, who can substantially contribute to integrating other models through collaborative exchanges.

It is also interesting to observe that, instead of understanding that a model needs to be satisfied by every data element of the respective dataset, it would also be possible also to allow the validity of the model not to be restricted to 0 (false) or 1 (true), but to depend on a graded merit figure such as the number of elements in the dataset that satisfy the model. The immediate implication of this is the loss of the bijective association between datasets and models. Now, instead of being underlain by a formal logic consistency, the modeling approach starts being understood as an optimization problem.

By blurring the frontier between data and models, the M^* approach paves the way to several interesting possibilities, including the definition of a paired algebra of dataset and models queries and manipulations. By ‘algebra’ we mean the ability to represent datasets or models as variables, or symbols that can be solved or interrelated through set operations (in the case of datasets) or logical equations (models). The association of a new dataset to a model corresponding to a combination of the previously available models provides not only a way to account for the respective dataset, but its intrinsic logical construction can also provide insights for computationally to decide if specific data elements satisfies that respective model. In addition, the obtained models may also provide indications about how the datasets were generated, sampled and obtained.

It is important to stress that the combinations of the existing datasets while searching for a match needs to be preceded by including in every existing dataset associated to a respective model all the elements in the new dataset that satisfies that respective model, and revising the overall consistency of the new state of the modeling framework. It is also possible to proceed gradually from new single data elements to the whole new dataset by progressively considering subsets of the latter having in-

creasing sizes. For simplicity’s sake, the present work will be restricted to updating the new individual elements and then the considering the whole new dataset.

Another possibility accounted for by the proposed framework consists in performing logical combinations between the model statements and then seeking which among the existing datasets can satisfy a new modeling statement. This same mechanism immediately provides the means for building programs for obtaining specific properties (determined by the model) to characterize and analyze respective datasets.

The above two possible approaches aiming at combining datasets or combining models are henceforth understood as being *data-driven* and *model driven*. These two ways of integrating information and knowledge seem to correspond to the main manners in which humans perform these two important activities. A third possibility consists of evaluating the pairs of datasets and models from the perspective of the current modeling framework.

As an example, consider that the original dataset ω_4 in Figure 7 becomes important enough to motivate the development of a respective model. We can approach the solution to this essential problem by searching for a combination between the datasets already explained by models that is identical to ω_4 . This can be done by checking between the result of set operations between the already instantiated datasets, such as:

$$?\omega_i \cap ?\omega_j \equiv \omega_4 \quad (1)$$

Sought datasets are henceforth represented with a preceding question mark, i.e. $?\omega$, while the \equiv symbol stands for being equivalent or identical.

In case it is verified that the datasets ω_2 and ω_7 satisfy Equation 1, by using set intersection, we can immediately derive the model of ω_4 as necessarily corresponding to:

$$m_4 = m_2 \wedge m_7 \quad (2)$$

The above described development of a model for the dataset ω_4 in Figure 7 is illustrated in Figure 8.

The developed framework also allows the features of m_4 to be immediately inherited from the features originally associated to m_2 and m_7 . As feature f_1 and f_3 were

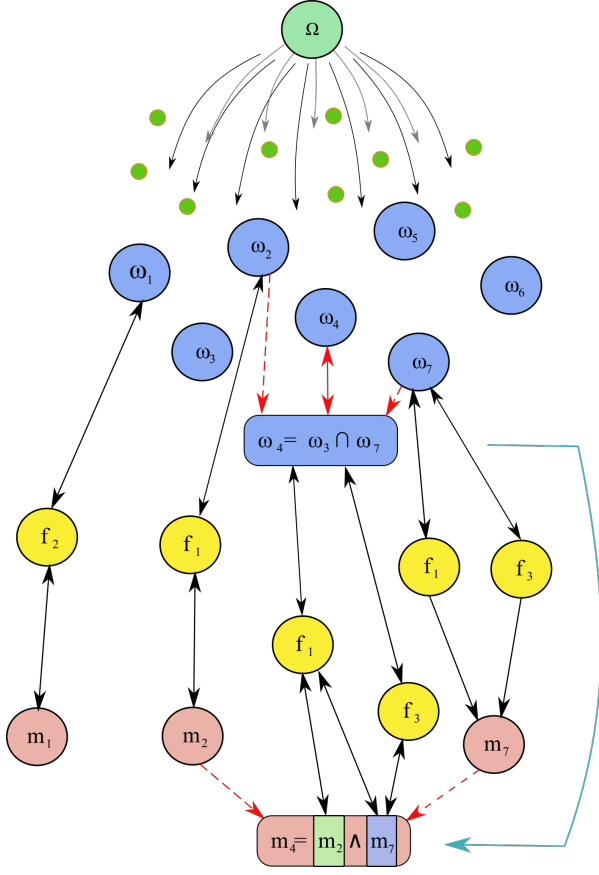


Figure 8: A new model is born. Having found that the dataset ω_4 corresponds identically to the intersection between ω_2 and ω_7 , and since both these two datasets are associated to respective models, it becomes possible to derive a model that satisfies the data elements in ω_4 . This can be done by defining the new model m_4 as having as input the features inherited from both ω_2 and ω_7 , which correspond to f_1 and f_3 . The model m_4 is immediately provided by the logical and operation between the models m_2 and m_7 .

required by both the original models, they also become pre-requisites of the new model m_4 . Observe that the different logical components of the latter model may receive different sets of features as input, as is the case for the present example. At the same time, it should be kept in mind that the bijective association between datasets and models depends critically on the choice of features as well as the current dataset and modeling framework.

Another example of the possibilities allowed by the suggested approach concerns algebraic equations as:

$$?m_i \cup ?m_j \equiv m_k \quad (3)$$

In other words, we would search in the current model framework M for a pair of models satisfying the above condition.

It is also possible to derive *hybrid* algebraic equations such as:

$$?\omega_i \cup \omega_j \equiv m_k \wedge ?m_p \quad (4)$$

This type of equation can be solved either by translating the dataset-based side (lefthand) into the respective model logical equation, or vice versa, and then applying the above mentioned procedures. In the former case, we would have:

$$?\omega_i \cup [\omega_j]^C \equiv m_k \wedge ?m_p \iff ?m_i \wedge \neg m_j \equiv m_k \wedge ?m_p$$

Alternatively, we could make:

$$?\omega_i \cup [\omega_j]^C \equiv m_k \wedge ?m_p \iff ?\omega_i \wedge [\omega_j]^C \equiv \omega_k \cap ?[\omega_p]^C$$

Interestingly, as models (as well as the respectively associated datasets) are progressively combined, a respective hierarchy is defined, as illustrated in Figure 9, which corresponds to the the last of the models shown in the list above.

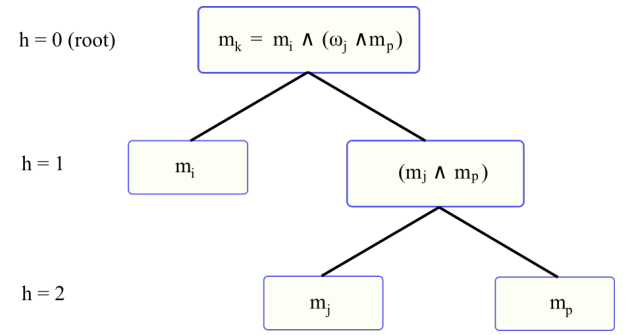


Figure 9: As models are developed and integrated by the proposed methodology, a respective hierarchy is progressively established as illustrated in this hypothetical example. The hierarchical levels are indicated as h , with $h = 0$ corresponding to the root of the tree representing the hierarchy.

As a consequence of the bijective association between datasets and models established by the M^* model, we immediately have that the above model hierarchy to be respectively reflected into the data hierarchy illustrated in Figure 10.

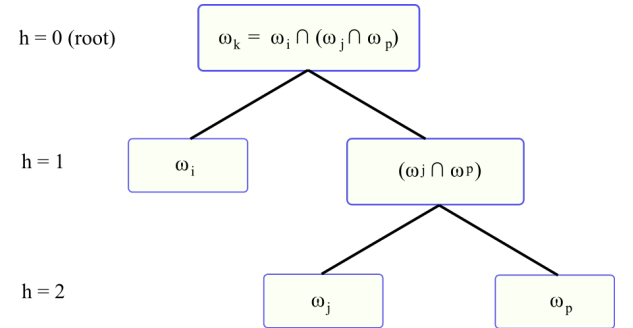


Figure 10: The dataset hierarchy respectively associated to the model hierarchy in Fig. 9 as a consequence of the bijective association between datasets and models established by the M^* approach.

Provided the number of datasets associated to respective models is not too large and that we are not aiming

at several hierarchical compositions of models, it is possible to search for the solution of problems by systematically checking every possible combination of the set/logic operations while progressively increasing the number of datasets or models.

It is interesting to observe that the level of *abstraction* in the modeling framework increases as we move from the leaves to the root of the respective tree. That is so because the understanding of composite models demands the understanding of the preceding models. At the same time, the level of *generalization* may increase, in case the combinations involve the union of sets, or decrease, as implied by intersections between the sets. As such, the proposed framework accounts for these two important paradigms, while also indicating that the abstraction increases in both these cases, as it is ultimately related to the complexity of the respective logical model.

The proposed M^* approach also relates to the critically important concept of *causality*. It is posited here that, at least as typically understood by humans, causality corresponds precisely to the bijective association established between a dataset of relevance and its respective model, and more specifically in the fact that *every* element of a data set satisfies, implies (or causes) the model. In other words, it is only the full presence of all the conditions of a model that can enable the model verification. Incidentally, observe that this possible definition of causality, which is after all a human concept just like complexity, implies the time sequentiality that is often used to characterize causality. After all, having all the data elements in the dataset representing the event of interest triggers the model (decision) conditions to be satisfied. The situations corresponding to incorrect identifications of causality would correspond to models only partially associated to an incorrect, though related model, which will otherwise lead to correlations between the observed events. Recall that in the proposed framework a same model may satisfy (imply or cause) two or more models.

In cases where the number of data elements satisfying each model is taken as a graded indication of model adherence, the existence of crossed connections, i.e. a data element satisfying more than one model, implies in respective correlations between the activations of models that are related to the datasets in a non-necessarily causal manner.

7 Case-Example: Binary Lattices

As a more concrete example of representing the construction of a model framework by using the concepts and techniques suggested so far, we consider each data element x_i to correspond to each of the possible instances of a bi-

nary lattice or array with dimension $L \times L$. By ‘binary’ it is meant that the lattice elements can only assume the values ‘0’ or ‘1’.

Therefore, we have Ω to correspond to every possible binary pattern on a lattice, while the datasets will correspond to subsets of the power set of Ω .

Given an integer value L , a total of $N_L = 2^{(L^2)}$ possible data elements are respectively defined. Figure 11 presents the set of all possible patterns for $N = 2$.

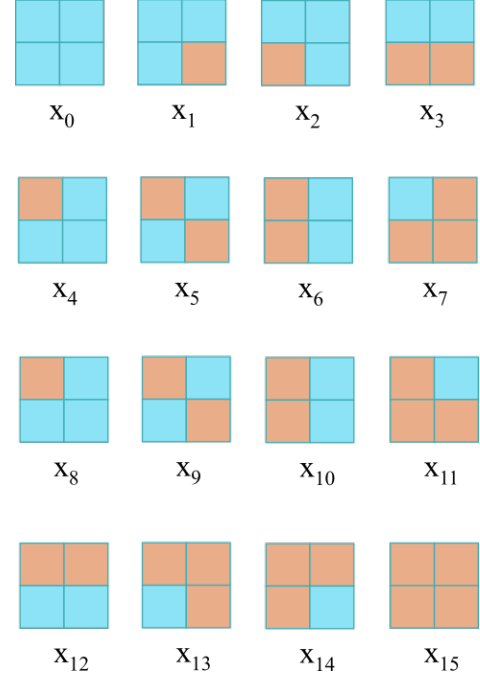


Figure 11: All the possible $2^{(2^2)} = 16$ data elements that can be derived from a binary lattice with dimension 2×2 . Zeros can be understood to correspond to the blue points, and ones to the brown points.

Observe that the number of patterns increases in a very steep, exponential manner. We shall adopt $L = 3$ for our first case example, therefore implying $N_L = 512$ possible data elements or basic patterns in Ω .

For simplicity’s sake, we will assume that E contains all possible data elements of Ω , but this is not a necessary requisite for the M^* approach.

Let’s assume that several datasets have already been singled out for their potential relevance and associated to respective models, as presented in Table 2.

This table includes the textual meaning of each model, the involved features and data structures, the formal condition representing the model, as well as the size of the dataset respectively satisfying each model. Recall that each of these models has a respectively associated dataset. Also, it should be realized that the choice of this initial modeling framework is, in principle, completely arbitrary, though the effective combination of models will also de-

m_i	textual description of the model	features/structures	decision	size
m_1	contains only an isolated point	number of points n	$n = 1$	9
m_2	contains 2 points	number of points n	$n = 2$	36
m_3	contains 3 points	number of points n	$n = 3$	84
m_4	contains 4 points	number of points n	$n = 4$	126
m_5	contains at least 3 points	number of points n	$n \geq 3$	466
m_6	conn. comp. contains $[1, 1]$	conn. comp. κ_k	$[1, 1] \in \{\kappa_k\}$	256
m_7	conn. comp. includes $[1, 1]$ and $[N, N]$	conn. comp. κ_k	$[1, 1], [N, N] \in \{\kappa_k\}$	88
m_8	the foreground set is thin	width w at each point	$w = 1$ or 2 for every point	291
m_9	opposite of m_8	width w at each point	$w > 1$ for at least one point	221

Table 2: A possible model framework M for the binary lattice case-example.

pend critically on these initial choices.

For simplicity's sake, each of the binary lattice elements, henceforth called a *point*, is here understood to be a square with four margins, two vertical and two horizontal. The points with value '0' are called *background* points while those equal to '1' are said to be *foreground* points. A point a in the binary lattice is said to be *adjacent* to another point b provided they share a vertical or horizontal margin. A *connected component* is a set of foreground points so that it is possible to move between any pair of their constituent points through adjacent margins. The *local width* of a set of foreground points is henceforth understood as the number of adjacent foreground points, i.e. neighbors. A binary lattice element is said to be thin whenever each of its foreground points has width 1 or 2 (e.g. [8]).

The models adopted in this example framework range from being very simple (e.g. m_1) to moderately complex (e.g. m_6, m_7, m_8), though this is a largely subjective classification. It is interesting to observe that us, humans, tend to have more difficulty in handling the complement of a model than its direct definition, as is the case $< 9 >$.

Let's now proceed to the dataset ω_{10} shown in Figure 12.

What is the model fully explaining this dataset? Can it be derived from the modeling framework in Table 2? This problem can be tackled by considering the several types of combinations of the existing datasets satisfying

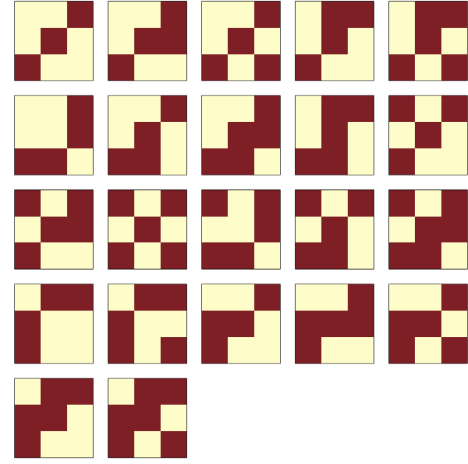


Figure 12: A puzzle. A dataset ω_{10} containing possible data elements derived from the 3×3 binary lattice needs to be associated to a model. Can it be derived from the model framework in Table 2? What is the property shared by these patterns?

algebraic constructions such as:

$$\begin{aligned}
& ?\omega_i \cup ?\omega_j \equiv \omega_{10} \\
& ?\omega_i \cap ?\omega_j \equiv \omega_{10} \\
& ?\omega_i - ?\omega_j \equiv \omega_{10} \\
& ?\omega_i \cup ?[\omega_j]^C \equiv \omega_{10} \\
& ?[\omega_i]^C \cup ?\omega_j \equiv \omega_{10} \\
& \dots \\
& [?m_i \cup ?m_j]^C \equiv \omega_{10} \\
& [?m_i \cap ?m_j]^C \equiv \omega_{10} \\
& (?m_i \cup ?m_j) \cup ?m_k \equiv \omega_{10} \\
& \dots
\end{aligned}$$

As it happens, it can be verified that the dataset of interest can be obtained through the combination $m_7 \cap m_8$ is identical to ω_{10} , therefore implying the following respective model:

$$\omega_{10} \equiv \omega_7 \cap \omega_8 \iff m_{10} \equiv m_7 \wedge m_8$$

It follows that, as the number of existing models (and datasets) increases, the higher the probability of finding a combination of those models that can explain a new dataset drawn from Ω . Recall that the compositions between the available models give rise to a respective hierarchical organization. Also, a new dataset can be explained by a completely new model not directly related to the existing ones, though perhaps sharing some of the adopted features.

Another interesting situation arises when a new model m is given regarding whether it will satisfy any of the existing dataset. This problem can be approached by trying to identify a logical combinations between the existing models that yield the new model, or by checking every existing dataset against the new model.

For instance, let the new model m_{11} be textually defined as “the dataset contains the shortest connected component comprising both the lattice elements $[1, 1]$ and $[N, N]$.” A possible first step is to try to translate this condition in terms of the available measurements and models. First, we select model $\langle 7 \rangle$, because it selects all data elements that contain at least a connected component containing $[1, 1]$ and $[N, N]$. Then, we take into account that the shortest possible path necessarily contains 3 points, which can be verified from model $\langle 2 \rangle$ while making $P = 3$. The sough model then can be obtained as:

$$m_{11} = m_2 \wedge m_7$$

The dataset satisfying this condition can be immediately identified as corresponding to the dataset associated to both ω_2 and ω_7 , corresponding to the 3-point diagonal between the points $[1, 1]$ and $[N, N]$. Observe that the logical construction of the obtained combined model allows us to directly obtain, through logic-computational means, the respective means for verifying the adherence of specific data elements.

8 The $M^{\langle \epsilon \rangle}$ Meta Model

So far, we have understood that all elements in all datasets satisfy the respectively associated model. However, it is possible that one or more elements currently in a given dataset do not satisfy the respective model. Possible causes for this include errors in the features determination, model inconsistencies implying the mathematical implementation not to correspond to the textual characterization of the model, or errors in storing and handling the datasets and/or models. These situations will be henceforth understood as corresponding to the presence of error or noise.

From the perspective of the present work, the most important consequence of errors and sampling is the loss of

the bijective association that is critical for the consistency of the M^* reference model, which leads to modeling, decision and classification errors.

In this section we describe an adaptation of the M^* model, here called $M^{\langle \epsilon \rangle}$, which can be considered for dealing with the above characterized situations, as well as for applications where only approximate verification of the conditions implied by the models are allowed. The underlying idea in all these cases is to adopt some effective means for quantifying the similarity between any two sets.

A possibility to cope with errors and sampling would be to relax the binary decision on the validity of a model that is characteristic of the M^* structure. This could be done by having by grading the degree of validity of a model.

In the present work, we will address the sampling and error limitations by the adoption of the following index that can express the similarity of two discrete datasets ω_i and ω_j :

$$\Lambda(\omega_i, \omega_j) = \frac{|\omega_i \cap \omega_j|}{|\omega_i \cup \omega_j|} \quad (5)$$

where the operator $|A|$ stands for the cardinality (or number of elements) of the set A . This index is known as Jaccard or Tanimoto index (e.g. [9]).

It can be verified that, conveniently, the above index is intrinsically normalized as $0 \leq \Lambda \leq 1$, so that it does not depend on the size of the sets.

Observe that the above mentioned graded validity of models can also be combined with the adoption of the Λ similarity for deciding on the associations between datasets.

In cases where each of the data elements $x_{i,j}$ of each dataset ω_i have been associated to non-negative *weights* $\alpha(x_{i,j})$ proportional to their relevance in the specific problem of interest, it is possible to adapt the similiarity index as:

$$\Lambda(\omega_i, \omega_j) = \frac{\sum_{x_k \in (\omega_i \cap \omega_j)} \alpha(x_k)}{\sum_{y_v \in (\omega_i \cup \omega_j)} \alpha(y_v)} \quad (6)$$

with $0 \leq \Lambda \leq 1$. Observe that this expression can immediately applied in case the data elements have been associated to probabilities, therefore paving the way to the probabilistic index to be described in Section 11.

As an example, let a set $A = \{1, 2, 3, 4, 5, 6, 7\}$, so that $|A| = 7$. Suppose a new set $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is to be compared with set A . This situation could be implied by obtaining a new version of a previous dataset, but incorporating by mistake the element ‘8’.

In this case, we would have $A \cap B = \{1, 2, 3, 4, 5, 6, 7\}$ and $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, therefore implying $|A \cap B| = 7$ and $|A \cup B| = 8$, from which we obtain:

$$\Lambda(A, B) = \frac{|A \cap B|}{|A \cup B|} = \frac{7}{8} = 0.875 \quad (7)$$

The index $\Lambda()$ therefore provides an interesting resource for checking if a new dataset could correspond to a noisy version of any of the existing datasets. This can be done by adopting a threshold T , and discarding any new dataset $\tilde{\omega}$ for which $\Lambda(\tilde{\omega}, \omega_i) \leq T$ for any of the existing sets ω_i .

Let us now illustrate the situation where a new version B of the existing set A is obtained while overlooking some elements, e.g. $B = \{1, 2, 3, 5, 7\}$. In this case, we would get:

$$\Lambda(A, B) = \frac{|A \cap B|}{|A \cup B|} = \frac{5}{7} = 0.7142 \quad (8)$$

As the obtained value is relatively high, it would suggest that the set B does not correspond to a new dataset and therefore can be merged into A or understood as being the same model.

However, in case $B = \{1, 5, 10, 12, 20\}$, we would obtain:

$$\Lambda(A, B) = \frac{|A \cap B|}{|A \cup B|} = \frac{2}{7} = 0.2857 \quad (9)$$

which is much smaller than in the previous case, suggesting that the set B does correspond to a new dataset.

As it will be presented in Section 11, the above similarity index can be adapted to datasets associated to respective probabilistic densities.

It should be observed that there are several other possible indices and methodologies that can be applied to deal with error and noise influencing data, features, and models. However, the above described alternative provides a particularly interesting approach especially given its conceptual and computational simplicity. In addition, though we currently discussed only possibilities for trying to avoid the incorporation of incorrect datasets, there are many other implications of incorrect or missing data and modeling that deserve to be further addressed at more length.

In brief, the $M^{<=>}$ meta model, as described here, can be simply understood as the M^* model that uses the adopted similarity index in order to identify the most likely combination of existing models while explaining a new dataset as well as to decide whether two datasets could be treated as being the same.

Also important to realize is that the above discussed errors and sampling imply in loosing the bijective association which is required for consistency in the reference M^* model, implying in respective modeling errors, classifications, and decisions.

9 Case Example: Elementary Number Theory

In order to illustrate the potential of the $M^{<=>}$ approach, a model of numeric sets taking into account the property of a number being a multiple of some radix is described in this section. This example involves new datasets that cannot be exactly explained by any of the models in the current modeling framework.

We start by defining $\Omega = \{2, 3, 4, \dots, 20\}$. The number 1 is omitted as it is a trivial divisor of any natural number.

The datasets ω_i , $i = 1, 2, \dots, 19$ will correspond to the multiples of $i + 1$ up to 20. Thus, $\omega_3 = \{3, 6, 9, 12, 15, 18\}$. The respective models are immediately derived from the respective multiplicity property. For instance, M_3 corresponds to “all the numbers smaller than 20 that are divisible by 3”. Therefore, the adopted modeling framework contains a total number of 19 pairs (ω_i, m_i) .

The first important point to be taken into account that these 19 models are by no means sufficient for explaining most of the possible new datasets that can be drawn from Ω . However, as we will see, the adoption of the Λ similarity allows a surprisingly good performance while being capable of providing interesting insights about possible explanations an interrelationships, even if no perfect combination can be found.

A modeling engine was implemented, using list manipulations in R, considering the following set operations (A and B are any of the existing datasets) shown together with the respective logical operations:

- $A \iff m_A$
- $A^C \iff \neg m_A$
- $A \cap B \iff m_A \wedge m_B$
- $(A \cap B)^C = A^C \cup B^C \iff \neg(m_A \wedge m_B)$ (De Morgan)
- $A^C \cap B = B - A \iff \neg m_A \wedge m_B$
- $A \cap B^C = A - B \iff m_A \wedge \neg m_B$
- $A \cup B \iff m_A \vee m_B$
- $(A \cup B)^C = A^C \cap B^C \iff \neg(m_A \vee m_B)$ (De Morgan)
- $A^C \cup B \iff \neg m_A \vee m_B$
- $A \cup B^C \iff m_A \vee \neg m_B$
- $(A^C \cap B) \cup (A \cap B^C) \iff m_A \oplus m_B$ (xor)
- $(A \cap C) \cup (A^C \cap B^C) \iff m_A \odot m_B$ (xnor)

Let's now consider the data-driven query relative to the new dataset $\omega = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

Only operations between two Boolean variables are considered for the sake of simplicity and also for keeping the results more accessible to human interpretation.

The engine found $\Lambda = 0.769$, respective to the model $M = M_2 \vee M_3$, which corresponds to the union of the multiples of 2 and 3. Though the similiarity index is not maximum, the provided explanation is still quite reasonable even if the given dataset cannot be fully expressed by the obtained combination (there are some values missing in ω).

Let's consider now another example, respective to the new dataset $\omega = \{8, 10, 12, 14\}$. We get $\Lambda = 0.5$ for two approximate solutions: $\omega_{12} \cup \omega_{14}$ and $(\omega_4 \cap (\omega_{10})^C) \cup ((\omega_4)^C \cap \omega_{10})$. Observe that, though all the numbers in this given dataset are multiples of 2, the set only contains 4 out of the 10 elements in ω_2 , so the result obtained is still fully compatible. Given that $\Lambda = 0.5$ can be considered too low, a new model would need to be defined for this dataset, as it cannot be approximated by combinations of those in the existing modeling framework. In this case, as observed in Section 5, it is also possible to associate to a restricted version of ω_2 , i.e. ω to $\omega_2 - (\omega_2 - \omega)$. Another possibility is to take into account a further feature, such as being comprised within a given minimum and maximum values.

Now, let's make $\omega = \{2, 3, 5, 7, 11, 13, 17, 19\}$. The result provided by the engine is $\Lambda = 0.875$, corresponding to two equally similar approximated solutions $(\omega_2 \cup \omega_9)^C$ as well as $(\omega_2 \cap \omega_{15}) \cup ((\omega_2)^C \cap (\omega_{15})^C)$. Interestingly, the prime numbers in the range from 2 to 20 could be well modeled in terms of the two combinations of sets, with a relatively high similarity index.

As another example, let's consider $\omega = \{3\}$. In this case we get $\Lambda = 1/3$, with 5 respective possible approximated combinations. This example illustrates the issue that it is difficult to express a small set as a pairwise combination of larger sets such as most of those in the existing modeling framework.

Let's now have the dataset $\omega = \{4, 8, 12, 16, 20\}$. By querying the described engine, we have two solutions for $\Lambda = 1$: ω_4 and $\omega_2 \cap \omega_4$, therefore capturing the fact that this last example dataset contains all elements between 1 and 20 that are both multiple of 2 and 4.

As a last example, we have $\omega = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$, which corresponds to the odd numbers between 2 and 20. The exact solution $[\omega_2]^C$ is provided by the engine.

As above illustrated, even though not ensuring full accuracy, the application of the $M^{<\epsilon>}$ can still provide valuable insights while understanding datasets and trying to find models for them.

10 Case Example: Verifying a Hierarchical Structure

In this section, we describe another illustration regarding how the M^* framework can be employed in order to verify a hierarchical structure. This example is intentionally simple, given its predominantly didactic objective. In addition, this examples illustrates only one of the many methods that can be applied for deriving or verifying a hierarchy.

Though the general problem of verifying or deriving hierarchies from data is particularly complex, requiring more sophisticated methodologies, the present example illustrates how some of the intrinsic aspects of the $M^{<\epsilon>}$ framework can be applied to this important problem.

Let's consider the hierarchical structure depicted in Figure 13.

The basic data elements and datasets that characterize a simple house have been organized into three successive hierarchical layers or levels, with the highest one corresponding to the topmost dataset, which is associated to the concept of *house*.

We start by taking the root, corresponding to the concept of *house*, as a reference.

First, we obtain the similarity index Λ values for each pair of nodes obtained by considering each of the elements from the three distinct hierarchical layers in the original tree. This yields non-null values only between pairs involving elements from the first and second levels, and second and third levels, with null values of similarity for pairs composed by elements from layers one and three. This result confirms the bipartition between adjacent layers, which is characteristic of hierarchical organization.

Then, we can check the type of relationship between the adjacent layers. Two situations will be considered here: (i) generalization, characterized by union of the datasets from the lower level layer; and (ii) specialization, characterized by intersection of the datasets from the lower to the higher adjacent layers. In order to do so, we obtain the similarity indices for each dataset in a given layer and pairwise union of the datasets in the lower hierarchy layer. Then, we repeat this verification but considering intersection, instead of union, between the datasets from the two adjacent layers.

In the case of the present example, we obtain higher values for union, corroborating that the hierarchical structure in this particular example has been obtained from the generalization of the datasets from lower to higher hierarchical levels.

Observe that other dataset or logical operations can be considered while analyzing or deriving hierarchies. For instance, it is possible to verify, by using set difference,

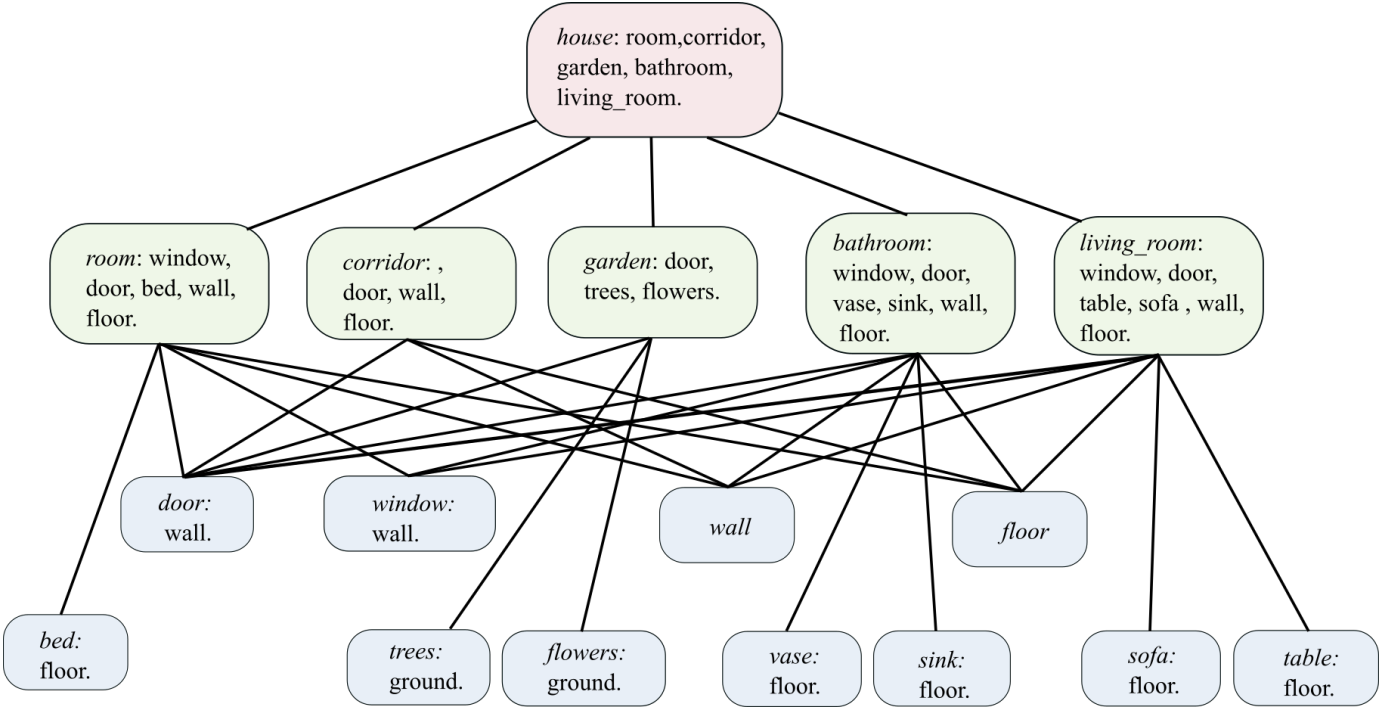


Figure 13: The hierarchical structure considered in the discussed case example, involving the description of a simple house. Three hierarchical levels, identified by light red, green and blue, can be identified.

in which manner the concepts between a same layer differ one another. A complementary type of analysis could involve using the *xor* and *xnor* logical operations. One particularly interesting possibility is to study, through the systematic application of all logical operations, how the presented hierarchy has been possibly defined and obtained, and to which a degree the identified sequences of logical operations are supported by the set operations between the respectively associated datasets.

By using the above identified concepts and methods, it is also possible to enhance a given hierarchical construction, e.g. by identifying links that are particularly weak and trying to complement the respective contents in the associated datasets.

Another interesting problem which tends to be more challenging than those already mentioned here concerns, given a set of datasets, to derive a respective hierarchy along generalizing or specification structures, an application that brings us close to the interesting subject of ontologies.

11 The $M^{(\sigma)}$ Stochastic Meta Model

There are several abstract and real-world situations in which the datasets are characterized by respective features that may extend continuously along the respective

axes in the feature space. Or, more importantly, there are cases in which Ω contains a huge number of elements, which tends to be the case for several real-world situations (e.g. the set of all possible butterflies). Mathematically, the feature-based representation of these sets can be properly obtained in terms of respective multivariate *probability densities* representing data elements distribution in the adopted feature space. At the same time as this probabilistic approach enables the consideration of many interesting problems, it also implies that the full consistency between data and model that is characteristic of the M^* to be undermined. This is to a great extent a consequence of the fact that the probability densities associated to specific models/categories often overlap one another. Each probability density is intrinsically associated to a respective *random variable*, or measurement.

This type of representation can be shown to provide virtually every statistical information that may be required regarding the dataset as described by the adopted features. For instance, the probability of observing all the data elements contained in a given subset of the feature space can be estimated in terms of the hyper-volume of the density taken on that region. The reader should not be put off by the seeming sophisticated adopted mathematical concepts, as the overall idea and principles are likely to be grasped with the help of the case-example provided in Section 12.

The representation of feature-based discrete, sampled

datasets also leads to the possibility of applying Bayesian decision (e.g. [8, 3, 5]) in order to decide what is the most likely category given a specific data element. This same approach also provides subsidies for estimating the probability of making incorrect decision. Even more importantly, the above outlined Bayesian decision method can be show to provide optimal results in the sense of minimizing the chances of making decision errors, However, this important property requires the availability of *exact* probability densities, but good results should be obtained for representative samples of data.

Because it is impossible to obtain an infinite number of samples allowing the complete characterization of these continuous variables, we need to resource to some suitable methodology capable of yielding satisfactory estimations in terms of *estimated* probability densities. The integration of this approach into the suggested M^* meta model leads to the $M^{<\sigma>}$ variant, capable of addressing situations characterized by incomplete data sampling. The remainder of this section presents a description of this approach.

Let ω_i , $i = 1, 2, N_\omega$, be datasets sampled from a universe Ω . Each of these datasets ω_i is chosen to be characterized in terms of a set of random variables (or features) f_j , $j = 1, 2, N_f$. In order to obtain a suitable probability density representing each of these datasets, it is possible to perform a kernel expansion (e.g. [5, 3, 10, 11]) on that set, with each individual data element being represented as a Dirac's delta function $\delta(\vec{f})$, $\vec{f} = [f_1 \ f_2 \ \dots \ f_{N_f}]$.

The gaussian kernel represents an interesting choice given its mathematical properties, but other kernels can be more suitable depending of the type of datasets and features of their original probability densities. The expansion itself can be performed by convolving (e.g. [10, 12]) the given dataset with a normalized version of the kernel. The estimated probability density obtained by kernel expansion of each dataset ω_i is henceforth expressed as $p_i(\vec{f})$.

Now, in order that each dataset be associated to a *support region* having finite hyper-area, we perform a thresholding operation on the respective estimated probability density. The resulting support hyper-region is henceforth indicated as ρ_i , which thus is necessarily a subset of the respective feature space.

The value of the threshold can be determined as corresponding to the situation in which the respectively obtained support region contains a fixed ratio χ of the possible elements (this is related to the concept of percentile). In practice, one is likely to choose large values of χ in order to retain a representative set, but other approaches may also be of interest.

Observe that the option of defining the support regions in terms of percentiles implies a more sparse dataset to

cover a larger area than a more compact counterpart. These situations may be addressed by taking into account also the density values associated to the support regions, reflecting the fact that the less dense portions of the support region will have smaller weight.

It is also necessary to derive the overall probability density as defined by all elements in all the existing datasets. This can be done by first obtaining the union of all available datasets, i.e. $\Gamma = \omega_1 \cup \omega_2 \cup \dots \cup \omega_{N_\omega}$ and then performing a kernel expansion possibly considering the same χ as adopted for estimating the other support regions. The estimated overall probability density is henceforth represented as $p(\vec{f})$.

The derivation of the support regions for each involved dataset has the immediate benefit of having finite area, immediately allowing them to be combined respectively to the set operations involved in the described M^* modeling approach.

We are now in a position to generalize the similarity index described in Section 8 in order to quantify the similarity between two sets A and B when represented in terms of probability densities, thus enabling the identification of the more likely model combination possibly explaining a new dataset.

Let A and B be two datasets described in terms of their respective probability densities *as well as* the associated support regions ρ_A and ρ_B as obtained for a chosen χ . The similarity between those two datasets can then be estimated as:

$$\Lambda(A, B) = \frac{\int_{\rho_A \cap \rho_B} p(\vec{f}) d\vec{f}}{\int_{\rho_A \cup \rho_B} p(\vec{f}) d\vec{f}} \quad (10)$$

We again have that $0 \leq \Lambda() \leq 1$. Observe that it is also possible to assign normalized weights to each of the intersection regions between the support densities in a new dataset corresponding to integration of the probability density within that same region. These weights can then be incorporated into the above equation, so that the portions of the support region of the new dataset that explains a smaller fraction of the overall data population have smaller influence on the decision.

It should be observed that the above described approach is still empirical, so that further formal validations should be developed. It is also possible to consider alternative methods for comparing between multidimensional distributions, such as those involving adaptations of the non-parametric Kolmogorov-Smirnov test (e.g. [13]) or the Jaccard index adapted to compare distributions and scalar fields [14]. In particular, the densities can be converted to multisets and then compared by using the Jaccard index [15].

The above obtained index provides a simple interesting manner for comparing a new dataset ω_k with combina-

tions of the existing datasets obtained through respective set operations, in an analogous manner as done in the M^* approach, but now also incorporating the estimated probability densities respectively associated to each dataset.

The association of the density distributions to limited area support regions also paves the way for applying mathematical morphology (e.g. [16]) operations on these sets, allowing the derivation of dilated and eroded versions, to name just a few possibilities, of the support and intersect regions, which can provide additional information about the shape and interrelationship of the datasets in the original feature space. For instance, it would be possible to dilate the regions of all existing groups and use the set difference between the result and the original support regions in order to obtain the border of the datasets.

Interestingly, the stochastic approach described in the present section paves the way to other important capabilities, including the possibility to obtain not only a likely combination of models explaining a new dataset, but also the quantification of how many elements of the latter relates to the each of the involved existing datasets. This possibility, as well as the overall stochastic model $M^{<\sigma>}$ described in this section, are further discussed and illustrated respectively to a specific real-world dataset. In a sense, this type of more complete description of a given dataset extends the concept of a dichotomic pattern recognition decision to a relatively more complete model providing additional information about how the new dataset explains and relates to the other existing datasets (and models).

12 Case-Example: The Iris Dataset

We now address a typical case of supervised pattern recognition by using the Iris dataset, which consists of 50 individual iris flowers from 3 species, each being characterized by $N_f = 4$ features. Thus, we have $N_x = 150$ individuals. We shall be restricted to features 2 and 3 in order to allow the feature space to be more easily visualized.

Figure 14(a) depicts the distribution of all the 150 individuals in the respective two-dimensional feature space, with the three categories being identified by respective colors.

The first step in our modeling approach consists of obtaining kernel expansions of the three groups of points which, in the case of the present example, is achieved by using a circularly symmetric gaussian as kernel assuming $\chi = 0.97$. The kernel expansion is then performed by convolving the original data elements in each group, which are represented as Dirac's deltas, with the normal-

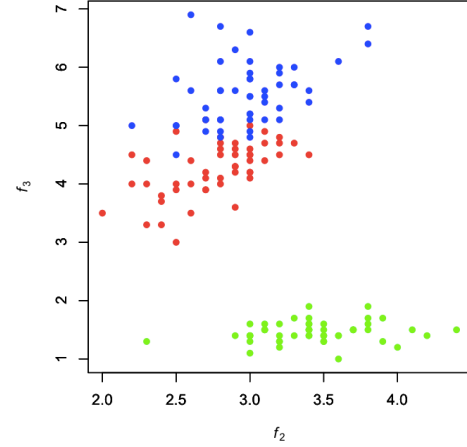


Figure 14: The three species in the iris dataset shown in a features space derived from the original features f_2 and f_3 : species 1 (green), species 2 (red), and species 3 (blue).

ized gaussian kernel. The adoption of a fixed percentile is reasonable given that the three datasets present a relatively similar sparsity. Observe also that, by varying the parameter χ multi-scale models of the datasets can be derived.

Figure 15(a) illustrates the result of the gaussian kernel expansion of each of the three datasets in Figure 14(b).

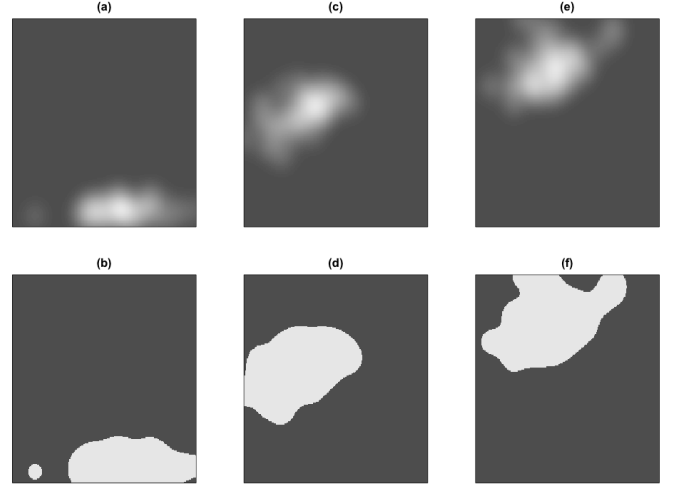


Figure 15: The densities and support regions obtained by non-parametric gaussian kernel expansion of the three types of flowers in the iris dataset: $p_1(\vec{f})$ (a) and $\rho_1(\vec{f})$ (b); $p_2(\vec{f})$ (c) and $\rho_2(\vec{f})$ (d); and $p_3(\vec{f})$ (e) and $\rho_3(\vec{f})$ (f). These results were obtained for $\chi = 0.97$.

Figure 15(b) presents the three decision regions obtained considering $\chi = 0.97$.

The initial modeling framework is assumed to contain the models indicated in Table 3.

The density $p(\vec{f})$ corresponding to the union of the densities associated with the three iris species is shown in Figure 16.

	<i>textual description of the model</i>	<i>features/structures</i>	<i>decision</i>	<i>size</i>
m_1	flowers belonging to species 1	f_2 and f_3	high similarity index for type 1	50
m_2	flowers belonging to species 2	f_2 and f_3	high similarity index for type 2	50
m_3	flowers belonging to species 3	f_2 and f_3	high similarity index for type 3	50

Table 3: The initial modeling framework M for the Iris case-example.

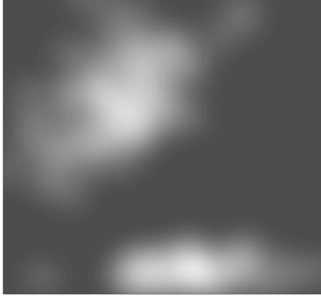


Figure 16: The density probability function corresponding to the union of the three iris species.

Let's now assume that a new dataset ω_4 becomes available, which is shown in Figure 17.

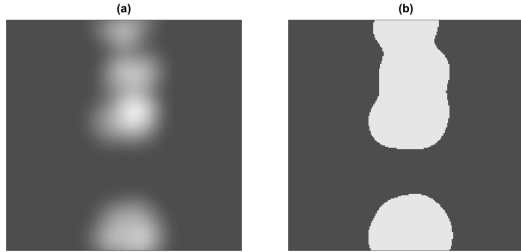


Figure 17: The density (a) and support region (b) of a new dataset.

This set is then also kernel expanded by the same gaussian as before, also using $\chi = 0.97$. The result is shown in Figure 17(b). Each new data element is then verified with respect to each of the three models, and updated respectively.

It is now possible to perform a search for possible combinations of the existing models that best explain the new dataset ω_4 . Among the several tried combinations, up to the second hierarchical level of composition of logical conditions, the following dataset was singled out as being the more likely to correspond to the new dataset ω_4 :

$\omega_1 \cup \omega_2 \cup \omega_3$	
• Overall similarity with ω_4 : $\Lambda = 0.375$	
→ 23 individuals related to 70.41 % of dataset 1	
→ 19 individuals related to 41.88 % of dataset 2	
→ 22 individuals related to 50.82 % of dataset 3	

Figure 18 depicts the three density probability functions associated to each of the three species after being clipped by the support region of the new dataset. The integration of these clipped functions provides the identification of the relationship between the new dataset and the three existing models.



Figure 18: The three probability density functions of the iris species clipped by the support region of the new dataset.

The number of individuals related to each of the existing datasets correspond to the number of data elements contained in the intersection between the support regions of the new dataset ω_4 and each of the other three existing datasets, which may to some extent overlap one another.

In the light of these results, the new dataset cannot be considered to be explainable by the union of the three original datasets corresponding to each of the three iris flower species. In addition, this new dataset seems to be more closely related to the iris type 1, though relatively similar relationships are observed also with the other two categories.

As such, an alternative explanatory model in the domain of plant science would need to be found or developed for this new dataset. In the case of this particular example, as the new dataset contains elements similarly related to each of the three original iris species, it could be conjectured that the new samples correspond to physical alterations, such as a disease or changing environmental

or genetic conditions, taking place on the flowers and implying the feature f_2 to shift in a similar manner for all the three species.

Several other insights can be derived from the obtained descriptions as in the previous example. For instance, in case a new dataset is found not to relate directly to any of the existing models while presenting a good relationship with the union of the respective complements, it may be associated to the borders between the clusters in a feature space. Such interstitial regions can provide valuable information for identifying effective separation regions in those spaces.

As an example, consider a new dataset whose respective density and support region is presented in Figure 19.

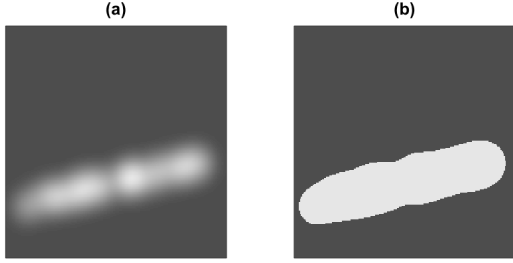


Figure 19: The density (a) and support region (b) of another new dataset.

The application of the described approach yields:

$\omega_1 \cup \omega_2 \cup \omega_3$	
• Overall similarity with ω_4 :	$\Lambda = 0.028$
→ 0 individuals related to 0 % of dataset 1	
→ 1 individuals related to 3.15 % of dataset 2	
→ 0 individuals related to 0 % of dataset 3	

Therefore, the second new data can be understood not to correspond to the model $\omega_1 \cup \omega_2 \cup \omega_3$. Unlike the previous example, however, the minute number of elements related to any of the existing datasets suggest that the second new dataset belongs to the borders or interstices between the existing data.

In case the new dataset corresponds to a single data element, the similarity of the $M^{<\sigma>}$ approach with Bayesian decision theory becomes more recognizable. Figure 20 illustrates the decision regions that are defined for the above data.

For each possible point (f_2, f_3) , the respective values are checked for every probability density associated to the existing datasets, and the label of the existing dataset yielding the largest probability density value is associ-

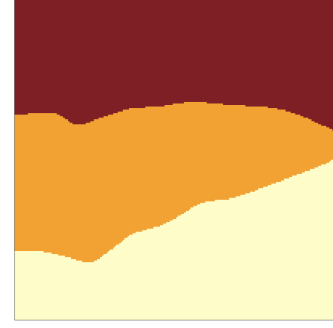


Figure 20: The decision regions obtained by considering Bayesian decision theory with respect to the Iris example. Observe the clipping of the overlap regions in shown in Fig. 15.

ated to that point. This example considers equiprobable datasets, otherwise the mass density of the classes also would need to be taken into account (e.g. [8, 5]). Then, each new data element can be classified as belonging to the dataset associated to the label indicated by its respective features in the decision region.

Thus, for relatively narrow gaussian kernel expansion, the incorporation of new data elements that precedes the checking for model combinations corresponds very nearly to the classical Bayesian decision theory. Though that approach naturally integrates resources that can provide information about not only data elements, but also datasets, as well as supplying information about the adherence of each element with respect to the several existing dataset other than the most likely one as well as the decision errors, it is felt that these possibilities are not often realized, perhaps as a consequence of the focus on dichotomic decision that is inherently motivated by the decision procedure.

Interestingly, it can be shown (see Figure 21) that the $M^{<\sigma>}$ meta model converges to the M^* reference model as the kernels become infinitesimal.

It should be realized that while the analysis of datasets in 2D can be performed visually by humans, the identification of most of the set combinations is typically difficult to be inferred in this manner, especially those involving combinations of set complements. The visualization of inference of set combinations in higher dimensional feature spaces is even more challenging to be performed by human operators, therefore providing even greater motivation for using automated methods such as the above developed.

Let's conclude this section by using the $M^{<\sigma>}$ approach to study the interrelationship between the three original iris dataset. The results obtained for each of these datasets are presented in the following:



Figure 21: The density obtained for the new dataset considering a much narrower gaussian kernel. Observe that the density tends to converge to the original points (Dirac's delta) as the width of the gaussian is progressively reduced, also implying the $M < \sigma >$ approach to converge to the M^* when the for infinitesimal gaussian width. The non-infinitesimal width adopted in the stochastic case is necessary in order to allow non-zero probabilities in the probability densities describing each dataset.

ω_1

- Overall similarity with ω_1 : $\Lambda = 0.2735$
- 35 individuals related to 96.4914 % of dataset 1
- 0 individuals related to 0 % of dataset 2
- 0 individuals related to 0 % of dataset 3

ω_2

- Overall similarity with ω_2 : $\Lambda = 0.4203$
- 0 individuals related to 0 % of dataset 1
- 45 individuals related to 96.597 % of dataset 2
- 16 individuals related to 41.393 % of dataset 3

ω_3

- Overall similarity with ω_3 : $\Lambda = 0.4709$
- 0 individuals related to 0 % of dataset 1
- 20 individuals related to 44.954 % of dataset 2
- 42 individuals related to 96.477 % of dataset 3

As expected, when taken separately, each of the regions resulted in relatively low values of Λ . At the same time, the well separated cluster defined by region 1 has been corroborated by the fact that no relationship has been found between this region and the others. This is not the

case with regions 2 and 3, which presented substantial mutual overlap.

13 And Now, to the Features Layer

We have so far focused on the bijective association between datasets and models as a distinguishing feature of the M^* model, understanding that the mapping between data elements and features was such as to ensure bijective representation of the datasets by respective sets of features. In this section we present which are the conditions that the mapping from data elements to features needs to present in order to allow bijective pairing between datasets and models.

Henceforth, each differently instantiated feature will be represented as f_j , independent of the feature types. By having a feature to be instantiated by a data element means that a given feature type was measured with respect to that data element, and the resulting value represented as f_j . Therefore, a same feature type may give rise to two or more instantiated features f_j and f_k provided the instantiations are mutually distinct.

Figure 22(a) illustrates several types of possible mappings from seven data elements $x_i \in X$ into seven several respective instantiated features f_j .

Given that we want the data elements to become bijectively associated with the feature instantiations, we revert the sense of every arrow, yielding the situation shown in Figure 22(b). The so merged groups of data elements and data features are represented by respective subsets shown as ellipses.

The case involving x_6 and x_7 deserves some particular attention. Here, we have that x_6 had been mapped into two features instantiations, namely f_6 and f_7 . If that was the only mapping received by these two features instantiations, we would have a situation completely analogous to that concerning x_1 and x_2 , implying both these data elements to be subsumed (merged) into a respective subset ω_1 . However, we have that also x_7 mapped to f_7 , implying that both x_6 and x_7 share this same feature.

As indicated in Figure 22(b), one solution for obtaining bijective pairing between these data elements and the respective features is to merge x_6 and x_7 into the set $\{x_6, x_7\}$, and the two features instantiations into the set $\{f_6, f_7\}$. Another possibility that may be considered depending on the specific interest and demands of each modeling approach in which x_6 and x_7 need to be kept separated, another possible solution for achieving bijective mapping would be to remove the mapping from x_6 to x_7 , especially in cases where this mapping is deemed not to be particularly relevant. This would lead to x_6 be-

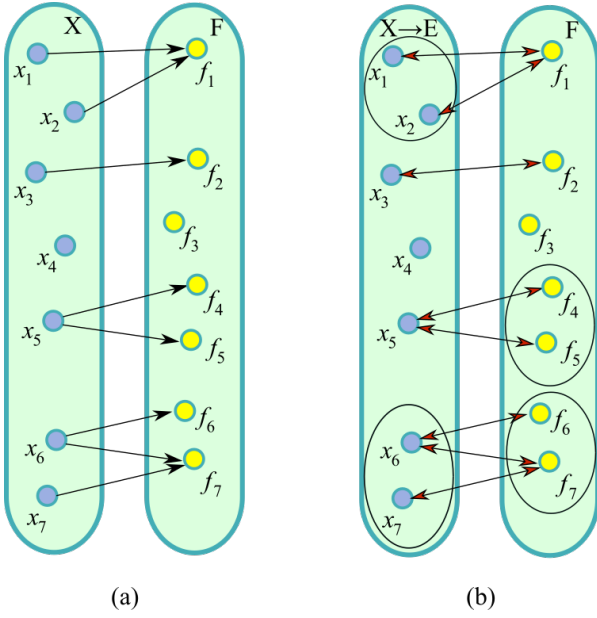


Figure 22: Mapping from data elements from the set X into the feature instantiations f_j in the domain F , illustrating several of the possible types of mappings that can be found (a). By incorporating the respective inverse for each mapping, leading to two-sided arrows, the data elements and features can be merged in the sense of forming a connected component. Valid datasets with respect to this configuration of X and F may only correspond to subsets of the resulting groups and separated data elements, i.e. $\{x_1, x_2\}$, x_3, x_4, x_5 and $\{x_6, x_7\}$.

ing bijectively associated to f_6 and x_7 with f_7 . A more complete approach would be to consider both possible solutions and then evaluate the two respectively obtained modeling frameworks.

The above example can also help to further illustrate the fact that the current data and feature domains have decisive influencing in making specific mappings of data elements into respective features. In Figure 22(a), we have that x_3 has been associated bijectively to f_2 . Let's now supposed a new data element x_8 is incorporated into X , and that it also maps into the feature instantiation f_2 . Now, f_2 it becomes impossible to distinguish between x_3 and x_8 , implying the bijective association to be transferred to $\{x_3, x_8\}$ and f_2 .

Now, we have that the merged data elements necessarily need to be comprised between the datasets ω existing in any current environment E of the M^* model, so as to maintain the overall consistency and therefore allow the bijective mapping between datasets and models. Observe that once the features have been associated in the above described manner, they become immaterial as far as the M^* structure and coherence is concerned. However, the relationship between features and data elements will need to be taken into account while incorporating new data and improving the choice of features.

Observe that the above method to identify the bijective associations between data elements and features can be immediately extended to handle bijective associations between *datasets* and features.

Figure 23 illustrates a possible bijective association between three datasets obtained from the data elements in Figure 22 and respective models.

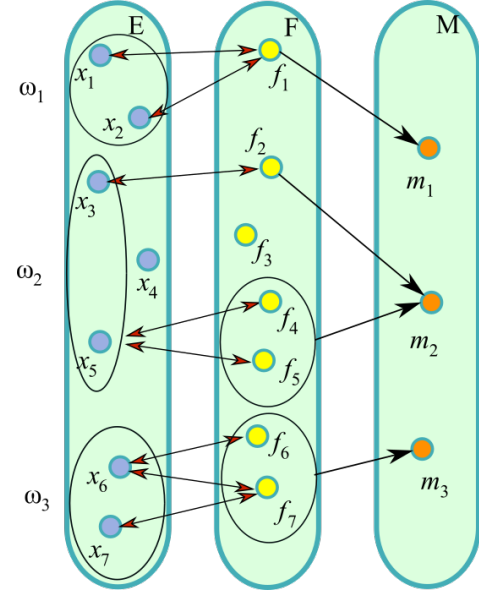


Figure 23: Example of possible bijective association between three datasets derived from Figure 22(b) and respective models in the modeling framework M , including the respective features also associated bijectively to those datasets.

Interestingly, observe that the above procedure ultimately allowed a bijective association to be established between subsets of data elements and subsets of features, except for the isolated data element and feature instantiation.

One point of particular interest here regards the fact that an association between data and model being bijective depends on the context of the data. This can be easily appreciated in terms of the following example. Let's say that we have a specific pair of glasses. In case that object is used only inside one's house (e.g. for reading), there is no need to specify this pair of glasses very completely, because there are no other similar objects to be distinguished from. This is by no means the case in other situations, as when we are in a crowd, which requires many more features to be specified. In brief, the bijective mapping of a dataset into a model also depends strongly on the respective environment E .

14 And How About Clustering?

It has already been observed in this work that there are two types of pattern recognition: supervised and unsupervised. As only the former has been considered so far in our approach, additional considerations can now be developed regarding the also important subject of unsupervised classification, which is also typically known as *clustering* (e.g. [5, 3, 17, 18, 19, 20]).

Generally speaking, clustering consists basically of finding separations or groupings between the respective distributions of data elements in the adopted feature space, as illustrated in Figure 24(a).

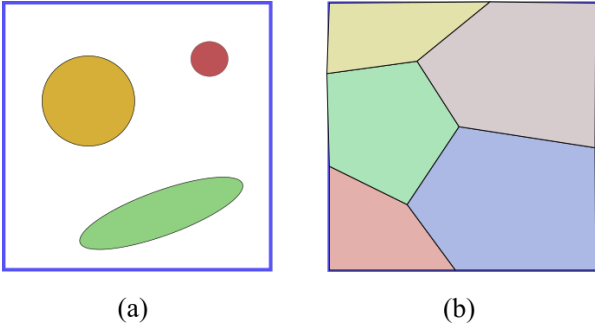


Figure 24: An example of clustered data containing 3 groups of points characterized by being well-delimited and separated (a), as well as another type of data separation which, though not having interstitial regions, still ensures all the data elements to be properly compartmentalized and classified (b).

In another related situation, also depicted as the situation (b) in Figure 24, the data elements are perfectly compartmentalized into respective categories or models, even though there are not interstices. The groups are also adjacent one another. This type of situation is rarely considered in clustering, because of being impossible to solve while considering only the spatial distribution of the data elements. As a matter of fact, the identification of clusters in cases where the datasets of interest are not well separated represents a substantial challenge in pattern recognition, because of the difficulties implied. At the same time, several real datasets tend to present overlaps and adjacencies, as is the case with the iris dataset in Figure 14.

Figure 24(b) depicts an interesting situation that helps to understand how the fact that a subset satisfies a model does not necessarily imply or relate to a cluster.

Here, Ω corresponds to a portion of \mathbb{R}^2 , so that the possible data elements are ordered pairs (x, y) , with $x, y \in \mathbb{R}$. The whole dataset ω delimited in the figure satisfies a well-defined model, namely $m : (x, y) | (x \geq 1) \wedge (x \leq 2) \wedge (y \geq 1) \wedge (y \leq 2)$. None of the points outside ω belongs to this model, so that we also have a perfect *partition* of Ω defined in terms of ω and $[\omega]^C$. Yet, despite all these

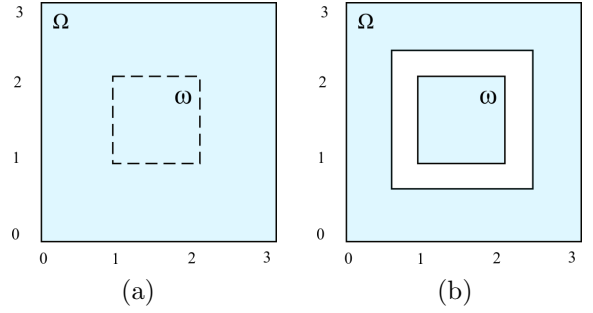


Figure 25: An example (a) of dataset ω which, though perfectly explained by the model $m : (x, y) | (x \geq 1) \wedge (x \leq 2) \wedge (y \geq 1) \wedge (y \leq 2)$, is not a cluster. In (b) we have a dataset ω that both is explained by this same model, but which is also a cluster. There is not need that a dataset corresponding to a model be a cluster, though clusters may motivate a dataset to be explained.

specific and distinct properties of ω , it is in absolutely no way clustered, or present interstitial regions with the remainder of the points in Ω . The main effect of the interstice is to call our attention on ω , motivating the explanation in terms of model that was already valid.

The above examples helps us to realize that both the datasets that are well-separated as well as the other types of compartmentalized datasets can be explained by models. In this sense, a dataset being a cluster is a property that would be independent of having a model associated to it or not.

A particularly interesting relationship between clustering and modeling relates to the fact that clustering provides one of the main means through which delimited datasets can acquire enough interest as to motivate respective explanation through modeling. The association of specific properties of interest (e.g. a well-separated group of bacteria capable of digesting some specific material) to a clustered dataset tends to make it even more likely to be modeled. It should be also observed that such distinguishing properties are primary candidates to be adopted as part of the feature for the specific characterization and modeling.

Another possibility worth considering is that, once a not necessarily well-separated dataset is identified as having special importance, it may become a more isolated group as a consequence of identification of more discriminative features, or even as a consequence of actions motivated by the need to separate the data, such as performing features transformations. For instance, a plant species that is initially little different from others, but present some interesting property, may be selectively breed to the point of being transformed into a more separated cluster. It is also possible to contemplate the situation in which actions are taken for reducing the cluster separation.

Yet another possible mechanism leading to the creation

of clusters is as follows. A single, or a few, data elements are observed to present a given property of interest. Efforts are then invested in identifying more elements satisfying that property, but without consideration of a control counterpart. As a consequence of this biased procedure, new data elements will be identified that present the desired property which will, as a consequence, yield a well-separated cluster, because the possible features that would possibly imply be adjacent in the selected feature having been filtered out.

15 Malleability of Datasets and Models

Given a structure represented as graph that is subjected to topological changes such as inclusion/removal of edges or nodes, it is possible to estimate the potential of this graph to undergo distinct successive changes. This can be done by using the recently introduced *malleability* measurement [21].

Because the M^* framework can be represented as a graph, it becomes interesting to characterize and compare the potential of distinct modeling frameworks in terms of their respective malleability.

The main problem when devising means to quantify the malleability of a graph or network concerns the fact that two or more of these structures (e.g. two separated instances of a network along time), as identified in terms of labels associated to the respective nodes, may actually correspond to the same topological structure, differing only with respect to the associated labelings, a property known as *isomorphism*.

As it is often very computationally expensive to decide whether two or more graphs are isomorphic, a viable alternative is to compare those networks after they have been mapped into a set of features. Remarkably, this mapping does not need to be bijective, provided we remain limited to comparing the networks from the perspective of the adopted features, which underlies the approach reported in [21].

Let γ be a graph, and let a specific manner to change this network be chosen. At a given time instant t , after the application of every possible instance of the considered change (e.g. removal of any of the possible edges), a total of D distinct networks are found to be derived from the initial configuration γ , each with a specific probability p_i . The malleability of this network can be calculated as:

$$\mathcal{M}_\gamma = e^{-\sum_{i=1}^D p_i \log_2(p_i)} = e^\eta \quad (11)$$

where η is the entropy of the probabilities p_i .

It is posited here that this index provides a good way to quantify the potential of a modeling framework to be

adapted for inclusion of new datasets and/or models. At the same time, it also supplies an objective means for characterizing the adaptivity and robustness of a given modeling framework.

16 Complexity

Complexity (e.g. [22, 23, 24, 25, 26, 27]) has remained a great challenge to be defined in an ample and yet accurate manner. This is all the most remarkable given the great importance this concept has achieved not only in scientific and technological fields, but virtually in all human activities.

Though continuing efforts have been made at grasping what complexity means, many of these have been conceived in order to address relatively specific problems by using respective concepts and approaches. A review of some of the main approaches to quantifying complexity can be found in [28].

More recently, an attempt has been made at obtaining a more comprehensive and flexible definition of complexity that would remain compatible with the way it is more generally understood by humans [28]. The underlying idea is to relate complexity to the *costs* of developing and operating/maintaining a model. Given that the concept of cost was conceived precisely to adapt to relative variations of specific resources availability and demands along time and space, the cost of a model seems to provide a particularly interesting perspective from which to approach the complexity involved in modeling.

We have already seen that several events and facts conspire to limit the modeling approach, such as described in the models reported here (Section 4). Some of the most relevant of those are now briefly discussed as indicators of complexity.

First, we have that real-world Ω universe sets tend to be extremely large, as it is characteristic even for specific types of plants and animals. The obtention and maintenance of these large datasets imply not only computational expenses, but also curation by experts.

The fact that the total number of subsets derivable from Ω corresponds to 2^{N_Ω} , where N is the number of elements in Ω , a combinatorial explosion soon takes place that makes unfeasible to consider systematic modeling approaches taking into account a substantial portion of the total possible number of datasets. Therefore, even in cases where the individuals are enumerable, and in absence of sampling and other types of error and noise, it becomes necessary to resource to optimization techniques capable of selecting particularly interesting subsets out of an extremely large number of possibilities. This implies substantial development and computational costs,

and it is poised to result in local minima, all of which contributes to making the modeling expenses considerable, accounting probably to many sources of complexity typically associated with modeling.

Then, we have situations in which the original data elements are too similar one another, implying a large number of features to be derived, some of which will probably imply in relatively high experimental costs. A related problem implying expenses and complexity regards the situations in which some of the individuals in E or Ω are rarely found.

To the above can be incorporated other several types of errors, noise, sampling and other limitations discussed in Section 4. We can therefore conclude that modeling at a more extensive and accurate level can become extremely expensive and complex in several ways and situations. As discussed in Section 4, creativity could be one of the best antidotes to complexity, allowing interesting results to be obtained even in challenging situations.

17 Complex Networks

With a history going back to the beginnings of humanity (e.g. maps), passing through the Königsberg bridges, graph theory, and sociological research, the subjects covered in the area of network science (e.g. [29, 30]), which focuses on complex networks, took off with studies of the Internet and the WWW. Briefly speaking, the subject of study of this area concerns graphs that present a topology that cannot be described in terms of one or few topological measurements such as the node degree (e.g. [31]). Therefore, a complex network would tend to present topological features markedly distinct from a regular graph or a stochastic counterpart such as a uniformly random network.

The remarkable success of this area, not only from the applied but also theoretical perspectives, resides greatly on the ability of graphs to represent virtually every discrete structure or phenomenon, also allowing for the fact that even continuous structures can be discretized to some resolution level. Another welcomed aspect highlighted by network science consists in its motivation for studies integrating the topology and dynamics of complex systems.

Given the above mentioned features of network science, it becomes particularly interesting to discuss, even if briefly, the relationship between complex networks and the meta modeling framework suggested in the present work. This is the subject of the present section which, however, shall focus on the issue that complex networks obtained from real or abstract datasets have been frequently derived while considering the *similarity* between the properties of the data to be explained (e.g. [32]). For instance,

in an informational network where the nodes stand for specific documents such as books, web pages, or works of art, the interconnection between these nodes is often performed while taking into account the content similarity, or overlap. Other examples involve networks constructed while considering the similarity between the features characterizing specific entities associated to nodes, such as in networks of living species, stars, shapes, etc. The similarity, or distance, between pairs of node is often represented in terms of weights associated to the respective edge.

A first important point here regards the fact that the very act of associating a node to an entity to be represented actually corresponds to identifying a model for that entity, which needs to be done in terms of a set of features. Then, these features can be compared, typically through similarity, while interconnecting the nodes.

While these approaches have great interest and potential, as already been demonstrated by the large number of well-succeeded applications (e.g. [33]), the often considered networks correspond only to one possible manner in which entities can be related, more specifically through similarities or distances. The framework describe in this work seems to allow several possibilities for extending the network-based representations. This corresponds, basically, to employing the several combinations of models obtained while deriving a network, involving several types of set operations and logical constructions. For instance, it is possible to connect two nodes corresponding to respective datasets with the node associated to the union, difference, etc., between these two datasets. Each of these connections could be identified by a respective label corresponding to the respectively applied set operation.

The integration between network science and the M^* framework can proceed mainly by considering both the individual data elements and respective datasets through the respective combinations allowed by the exact or approximated combination of datasets and models. In approximated cases, it may be of particular interest to compare networks representing an exact modeling framework with the available datasets that can be approximately explained by each of the theoretical models. Other possibilities already hinted in this work are to derive bipartite networks from the association between data elements and respective models, as well as considering the hierarchical constructs resulting from combinations of data or models as networks, to which similarity links may also be added.

Though the possibilities are many to be identified and discussed here, it is hoped that the above discussion may motivate further related analysis and developments.

18 Collaborative Research

The M^* meta model provides several interesting subsidies that can be employed to obtain insights about the characteristics and challenges in collaborative research. Of particular importance here is the requirement in the M^* metal model of keeping full compatibility and consistence between datasets and models.

Science has largely relied on the integration of two complementary approaches: individual and collective. In the former case, we have a single scientist, possibly with the assistance of a team, working on specific problems. The latter approach is characterized by more ample collaborative initiatives involving big projects, regular meetings, and, more recently, WWW-based resources.

While the case of individual research can be directly related to the the meta models suggested in the present work, the collaborative counterpart requires more analyses. In fact, it should be observed that a fully individual research initiative is virtually impossible, as one needs to learn and to communicate concepts and results.

Figure 26 illustrates a highly simplified situation involving several agents like that in Figure 1 that can collaborate one another through the represented network of information exchange.

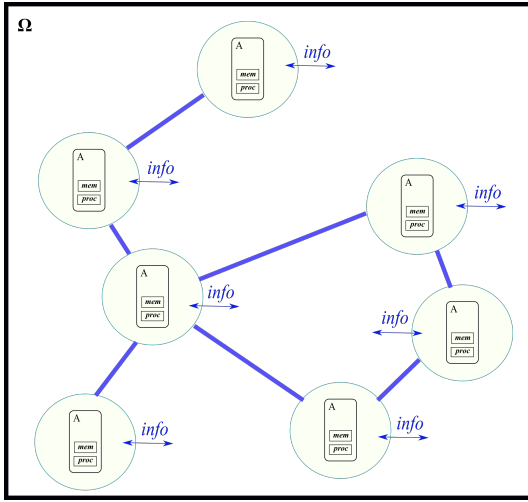


Figure 26: A highly abstracted and simplified model of interaction between agents (living beings or machines) that develop models. Issues of particular relevance for collaborative research regards how well integrated the agents are, which depends on the network topology, as well as the importance of keeping consistency between communication, datasets and models not only at the individual agent level, but also among all the modelers. Two possible ways to implement the latter include through shared external resources as dabatases, or by implementing continuing exchanges of information between the agents.

As each researcher gathers new data and develop respective models, it becomes important to *communicate* these findings as wide as possible through the network,

so that the results can be further validated and the other modeling frameworks can be updated and kept consistent. This requires a *shared, standardized or translatable* (e.g. via meta models) body of datasets and models. At the same time, it is essential that the same features are validated and adopted by all or most researchers. The larger the modeling framework, more possibilities can be tried while combining and integrating the respective models, though adding substantially to the overall complexity.

Of particular interest becomes the possibility to construct shared *databases* integrating all existing datasets and respective models, as well as employing *automated* means for implementing or assisting the modeling activity. The latter is of critical interest because the typical amount of information and knowledge currently necessary for many areas by far exceeds the cognitive and memory capabilities of any human being. Special attention should be also given to establishing common data and modeling formats and representations, especially concerning the identification of suitable data structures to be adopted in each specific situation or generalized as shared resources.

Other challenges in collaborative research involves how to cope with the several types of modeling errors discussed in this work, to which can be added errors and limitations of the communication through the existing network. Data and model *curation*, possibly assisted by automated means, could contribute to achieving reasonable levels of data and model quality. As a matter of fact, it is also important to continuously keep and expand the communicating resources.

19 Deep Learning

Similarly to network science (e.g. [29, 30, 31]), deep learning (e.g. [34]) has achieved substantial acceptance and success in a relatively short period of time, while also relying on approaches going back to the 19th century, a great deal of which related to the neuronal network paradigm.

The success of deep learning stems mainly from the fact that it paved the way to solving many problems that had remained as big challenges for pattern recognition. This has been achieved thanks to several factors (e.g. [34]), including the consideration of vast amounts of data and computing resources, as well as the development of new and creative concepts and methods. Most deep learning systems have the neuronal elements arranged in a several sequential and/or parallel layers containing a vast number of components.

A typical deep learning system can be understood to involve a vast number of basic processing elements, analogous to neurons (e.g. [35, 36]), that are successively applied (often through convolution) onto the incoming data

in order to derive valuable features and perform successful pattern recognition.

Following the approach reported in this work, it should have become evident that the development of formal M^* models requires strict maintenance of dataset and model consistency not only within themselves, but also one with the other, while also involving the ample consideration of every data element in the environment E at all times. For proper operation of the described modeling approach, every data element and model would need to be considered and related along the whole activity of modeling. This fact may well be related to the critical importance of taking very large training dataset and processing resources as it is characteristically found in deep learning.

Though remarkably successful in many applications, deep learning also has its respective challenges, including the relative difficulty in inferring the rules through which the solutions have been obtained, or translating the obtained trained parameters into formulations that can be more easily communicated to humans.

Given that the M^* and other related approaches described in the present work provide a relatively formal description of how datasets can be mapped into models, they may pave the way for identifying related methods for inferring the learned classification rules and translating them into more tangible statements. This could be done, for instance, by trying to associate formal or textual models to some of the datasets that have been assigned to categories by a respective deep learning system and then trying to explain other, more complex datasets, in terms of logical combinations between the identified datasets. The $M^{<\sigma>}$ stochastic variation of the M^* approach may be of particular related interest, as it is directly related to the decision regions normally associated to the basic deep learning processing elements.

It may also be possible to conceive manners of integrating the M^* framework within a deep learning system, so that the recognition of the input datasets can be directly accompanied by the identification of possible respective models. Another interesting possibility would be to try to adapt the impressive hardware resources developed for gaming and deep learning to perform the basic M^* operations and manipulations in a faster manner.

20 Creativity

As complexity and other words that have received great attention, *creativity* has also proven to be a challenge to being defined and characterized. Here, we will adopt the approach described in [37, 2], more specifically that creativity corresponds to manners of achieving effective results with relatively great efficiency, little cost, and great

innovation. It is from this perspective that we will briefly discuss how modeling, especially as approached in the M^* initiative, is related to creativity.

One of the main ways in which creativity can be achieved consists in seeking for analogies or metaphors between two or more problems that, though belonging to different areas, present some interesting similarities and analogies. An extremely simple illustration of this type of creative association is the pairing of numeric sequences such as 1, 2, 3, ..., with successive letters as a, b, c, \dots

A particularly interesting point is that the M^* modeling framework is critically dependent on establishing an effective bridge, through a strict bijective association, between set operations in the dataset domain and logical manipulations in the modeling domain. In addition, by providing possible logical and mathematical explanations that can be eventually translated into textual statements defining a model, the proposed framework contributes to making those dataset more accessible and tangible to those interested in their respective analysis and modeling.

In addition, engines analogous to those illustrated in this work may be employed as means of providing insights about possible relationships between models and datasets, as well as clues on how to combine or develop new models capable of explaining new datasets.

Following similar reasonings, the consideration of concepts and methods from a large number of areas adopted in the development of the M^* framework also contributes to identifying possible creative analogies between their respective properties, challenges and advantages.

Additional aspects of the M^* approach that may favor creativity include the establishment of relationships and similarities between the several datasets, which may be related to different fields. Thus, the basic set and logical operations underlying the M^* operation can be understood to be directly related to potential creative associations between different datasets, areas, features, and types of models. The fact that several types of features and models may be incorporated into the suggested meta models also contribute to provide grounds for creative investigation.

Another not so directly identifiable aspect concerns the fact that the M^* approach evidences the breathtaking combinatorial complexity involved in model building for relatively large, or even moderate, Ω sizes. This huge complexity can also be understood as contributing more space and degrees of freedom while seeking for creative approaches and solutions. Interestingly, complexity and creativity seems to be in a sense intrinsically connected and interdependent, even though they often oppose one another.

21 Concluding Remarks

And so we have reached the conclusion of the present work. It has been a relatively long development, as implied by its wide main objectives of taking into account many concepts and areas as the main subsidy for developing a putative meta modeling approach that could provide some insights about model building, decision taking, and pattern recognition, among other possibilities.

We started by discussing what we called the *informational schism* that is unavoidably established between the real world and any modeling agent, be it a living being or a machine. As it has been argued, the appearance of these agents was only have been allowed by the creative incorporation of modeling abilities capable of providing effective means for interacting with the respective environment. This modeling ability is particularly critical because it ultimately provides the means for taking effective decisions on subsequent actions based on previous experiences and the consideration of current environmental conditions.

Because of the central role of model building in so many areas, including pattern recognition, it becomes interesting to develop respective abstract models capable of providing some insights and resources for modeling. The main requirements that such a meta model should have were then identified and listed, including many constraints and sought properties.

The critically important task of mapping data into models was then approached, with special attention being focused on the need to preserve as much information as possible, which was shown to be only fully possible in case an bijective association is established between the existing datasets and the respective models. This relationships can be achieved by always ensuring that every element in a dataset is satisfied by the respective model, and vice-versa, therefore implying in a respective bijective mapping. It has also been shown that the current dataset environment, as well as the choice of features, also contribute in defining whether a given association between dataset and model is bijective or not.

At the same time, the generalization provided by each model in explaining all the elements in the associated dataset, which requires a non-injective mapping, was accommodated in the fact that this non-bijective relationship is maintained at the level of (*dataset, model*) associations. The incorporation of parameters and features, two important components of model building and pattern recognition, was also discussed and addressed.

Subsequently, we identified and briefly considered some of the main sources of limitations in achieving a fully complete and precise model, which include the presence of noise, sampling, and errors. The characteristics and ef-

fects of these limitations were then discussed respectively to the main components and actions involved in model building.

Having thus obtained subsidies from the presented and discussed several points involved in modeling, decision making and pattern recognition, we started an approach to developing a relatively formal meta modeling framework, which was called M^* .

This framework provides several interesting features that emanate from the strict bijective association established between data and models, including the definition of a bridge between these two worlds and the derivation of a paired algebra of datasets and models which can be employed to find models for new datasets through the logical combination of model statements as well as in terms of set operations between the existing datasets. The M^* approach was then illustrated respectively to a case-example involving datasets composed of patterns derived from binary lattices. In fact, despite its seemingly strict requirements, there are several problems characterized by relatively small amounts of discrete data that can be approached by using the M^* framework.

The requirements underlying the M^* approach were then progressively relaxed as the modeling approach was extended to cope with some of the severe limitations that are characteristic of pattern recognition and scientific modeling, including noise, errors and sampling. This led to the $M^{<\epsilon>}$ and $M^{<\sigma>}$ meta models, the former accounting for comparing data in presence of error or sampling, which was illustrated respectively to elementary number theory and verification of hierarchical structures, while the latter incorporating means for dealing with frequently necessary stochastic description of datasets in terms of probability densities, subsequently illustrated for a database related to the iris flowers.

Though we have seen that pattern recognition corresponds to a kind of modeling, it often has the specific characteristic in which the condition to be satisfied corresponds to the pertinence of the respective dataset with a given category. Oftentimes, but not always, the result of pattern recognition does not incorporate a more complete description of why the respective dataset has been decided to be assigned to a certain class. In addition, the data environment E tends to be constrained in practical pattern recognition problems, while in scientific modeling E is usually assumed to extend as much as possible toward the whole physical world. Nevertheless, pattern recognition remains an instantiation of model building.

In addition to providing insights about the intricacies of modeling, the suggested frameworks may also be used to derive practical methods and software engines for automated or assisted scientific modeling and pattern recognition, among other possibilities.

At the same time, the presented developments also emphasize the need to keep data and models as much as possible consistent and integrated, which poses some specific challenges regarding formats, data integrity, validation, among other issues.

The important related subjects of clustering, complexity, collaborative research, deep learning, and creativity were then considered and discussed in terms of several of the concepts and insights provided by the reported modeling framework.

The many implications and possibilities allowed by the presented concepts and methods pave the way to a substantial number of possible future developments. These include, but are not limited to, further extending the family of models derived from the reference M^* framework so as to be able to address additional constraints, developing effective concepts and methods that can be used for implementing kernel expansion in higher dimensional feature spaces, design practical engines for application of the described models, integrate the latter with and within deep learning concepts and implementations, and consider other important activities that are also related to model building (e.g. planning, diagnoses, learning, recommendation, etc.). A particularly fundamental remaining question is if real-world entities have some precisely well-defined respective models that are completely independent of humans and, if so, how these models could be somehow inferred.

Going back to the initial problem of how living beings and other modeling agents including humans and machines may have overcome the so-called information schism, the concepts and possibilities discussed in Section 18 can be understood to have provide respective insights. More specifically, the information schism seems to have been circumvented by creative modeling frameworks which may well be directly related to those described in the present work, especially in the sense of providing means for progressive adaptation to the environment demands, and by establishing communications and collaborations between the involved modeling agents. These insights may also extend to smaller scales, as at the molecular or cellular level, such as regarding the onset of multicellular organisms which could be understood as a interconnected body of individual specialized agents exchanging mass, energy and information.

All in all, it is felt that the developed modeling approach has potential for satisfying at least partially many of the requisites listed in Section 2. At the same time, it is important to keep in mind that several of the suggested concepts and models are still subject of further formalization, validation, and extensions. Perhaps one of the main results of the described developments ultimately resides in the fact that while it is interesting to find out

if a given dataset belongs to a specific category, it may be even more interesting to have a complete objective description, in the form of a respective model, providing possible explanations of why this takes place. Another critical aspect of the M^* framework is that, though it can be applied even at the level of individual data elements, it emphasizes the consideration of *collections* of elements as providing more significant grounds for strong model association.

We conclude by presenting in Figure 27 a graph abstract illustrating the main concepts addressed in the present work, as well as some of their most relevant interconnections.

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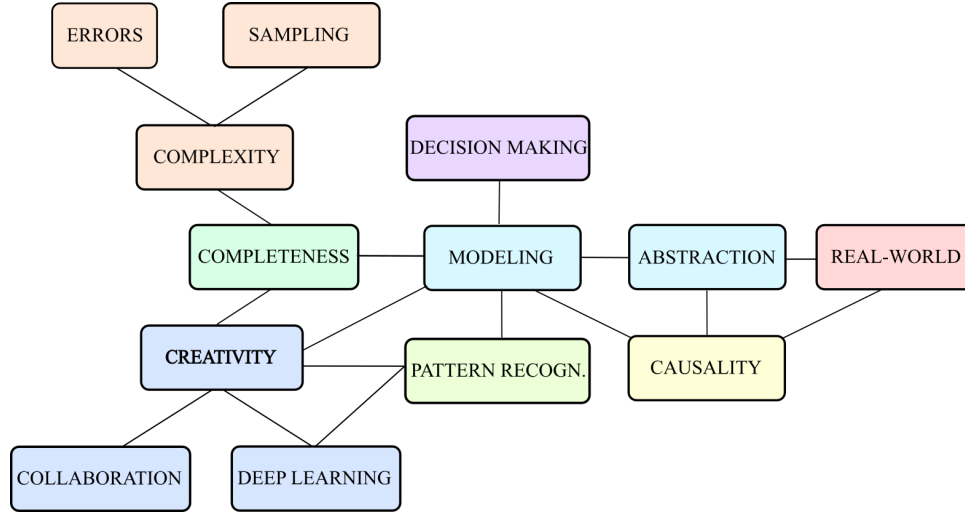


Figure 27: Graph abstract interrelating the main subjects of this work. The main concepts and areas covered here, as well as some of their most important interconnections. The main issues associated the modeling challenges are shown in orange, while some possible concepts that can contribute for solving those issues are represented in light blue.

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